This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`

STANFORD UNIVERSITY LIBRARIES

NEW SERIES

OF THE

MATHEMATICAL

REPOSITORY.

BY THOMAS LEYBOURN,
Of the Royal Military College.

VOL III.

London:

PRINTED AND SOLD BY W. GLENDINNING, NO. 25,
HATTON GARDEN.

1814.

CONTENTS

OF THE

THIRD VOLUME.

PART I.

MATHEMATICAL QUESTIONS.

Art, Page.
I Solutions to Questions proposed in No. VIII. viz. 241 to 270... 1
II. Solutions to Questions proposed in No. IX. viz. 271 to 290 .. 53
III. Solutions to Questions proposed in No. X. viz. 291 to 310...105
IV. Solutions to Questions proposed in No. XI. viz. 311 to 330...169
Notices relating to Mathematics, pages 49, 97.

PART II.

ORIGINAL ESSAYS.

Art. Page.
I. Mr. **Gough** on Polygonal Numbers........................... 1
II. Mr. Knight on the Binomial Theorem, and the Logarithmic Series.. 8
III. An illustration of the 47th Prop. of the 2d Book of the Principia, by A. B.. 10
IV. A curious indeterminate problem, by Mr. **Cunliffe**............ 13
V. Solutions to a curious problem in dynamics, by J. L............ 15
VI. Mr. **Knight** on the Expansion of a formula connected with Physical Astronomy.. 32
VII. Mr. **Knight** on the sine and cosine of Multiple arcs........ 34
VII. Mr. **Noble** on Magic Squares.................................. 37
IX. Baron Maseres on the mutual attraction of all the particles of matter.. 38
X. Mr. **Barlow** on Polygonal Numbers............................. 44
XI. Mr. **Barlow** on the irreducible Case of Cubic Equations........ 46
XII. New Properties of the Conic Sections, by Mr J.F.W. **Herschel** 58
XIII. Indeterminate Problems, by Mr. **Cunliffe**..................... 60
XIV. Mr. **Cunliffe** on the Ellipse and Hyperbola 66
XV. Mr. **Barlow** on the approximation of the roots of Equations. 97
XVI. Properties of the right-angled triangle, by Mr. **Bransby**........ 71
XVII. Mathematical Scraps by Mr. **White**............................. 81

PART III.

MATHEMATICAL MEMOIRS EXTRACTED FROM WORKS OF EMINENCE.

Art. Page.
I. Legendre's Memoir on Elliptic Transcendentals concluded.... 1 to 45
II. A formula for the length of a circular arc, by Euler.............. 46

Cambridge Problems, for 1811, 1812 and 1813, pages 1 to 38.

Directions for the Binder.

I. The Binder is desired to observe, that the Vol. consists of different sets of signatures. The first set A, B, C, D, contains the questions, and is to be placed at the beginning of the Vol. to serve as a table of contents to the first part.

II. The first, second, and third parts of the Vol. must follow in their order. The two sets of questions of Vol. 4. (viz. the signatures A and B) which are stitched up with the numbers forming the third vol. must be preserved apart till the fourth vol. is completed.

III. The Cambridge Problems to be placed after the third part.

The Plates may be placed as follows.

PART I.

I.	page 48
II.	96
III.	168
IV.	206

PART II.

A	32
B	80

Plate A′ to be placed at the end of the Vol.

THE

MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS,

To be answered in Number XII.

I. QUESTION, 291, *by Mr* GERMAN BUXTON, *London.*

To find two numbers whose difference is a square; and if any number (n) be added to each, or to their sum, they will make three other squares.

II. QUESTION 292, *by Mr* CUNLIFFE, *Royal Military College.*

If a quadrantal arc of a circle, radius 1, is divided into any four parts, the sum of the products of the tangents of every two of these parts is equal to 1 + the continued product of all the four tangents: Required the proof?

III. QUESTION 293, *by Mr.* DAWES, *Birmingham.*

In what latitude, on the longest day, will the shadow of a point of a gnomon, describe a right-angled hyperbola upon an horizontal plane?

IV. QUESTION 294, *by* G.V.

Find three square numbers in arithmetical progression, the sum of the square roots of which shall be a cube.

V. QUESTION 295, *by Mr.* W. WALLACE, *R. M. College.*

THEOREM. In any conic section, if normals be drawn at the extremities of a chord passing through the focus so as to intersect one another, the sum of the squares of the parts of the normals between the curve and their intersection shall have a given ratio to the square of the chord. Required the demonstration?

VI. QUESTION 296, *by Mr.* CUNLIFFE.

Suppose VmR a curve in a vertical plane, and P a given point in its axis; to find its nature, so that the pressure of a given weight m, attached to a string mP, fastened at P, and resting on the curve, may be every where the same.

VII. QUESTION 297, *by* G.V.

Required the nature of a curve, such, that a tangent being drawn from any point in the curve to meet the axis, the distance of the point of contact from a given point in the axis shall have a given ratio to the segment of the axis between that point and the tangent.

VIII QUESTION 298, *by Mr.* LOWRY, *R. M. College.*

To determine a curve that shall cut an indefinite number of curves of the same given species, having a common axis, so that all the tangents drawn to the given curves at the points of intersection may tend to the same given point.

IX. QUESTION 299, *by* HERMODORUS.

Given the vanishing line its centre and distance of an original plane, and the projection of a straight line, and of a point in that plane, through that point to draw a straight line so that the angle made by the two straight lines shall be equal to that made by their originals.

X. QUESTION 300, *by* AMICUS.

To find curves that have the following property, *viz.* that if P be a given point in the straight line APB given by position, and any straight line whatever be drawn through P to meet the curve in two points, and tangents be drawn at these points to meet the line AB in E and F, the segments PE, PF shall always be equal.

XI. QUESTION 301, *by* H.

Supposing a particle of matter at rest any where upon the surface of a solid of revolution which revolves about its axis in a given time, and that the particle is attracted towards the centre with a force which is as some determinate function of its distance from the centre: What ought to be the form of the solid?

XII. QUESTION 302, *by* G. V.

THEOREM. The square of any prime number of the form $4n + 1$ is of the form $a^2 + 25b^2$. Required the proof?

XIII. QUESTION 303, *by Mr.* CUNLIFFE.

Suppose two projectiles thrown from the same place with equal velocities, in the same vertical plane; to determine the angles of elevation, so as to include the greatest space within the curves described in their flight, the horizontal range being the same in both.

XIV. QUESTION 304, *by* EDINBURGENSIS.

Find the fluxional equation of the *curve of pursuit* when the body pursued moves in any curve of a given nature.

XV. QUESTION 305, *by* L.

To find (from elementary principles) how a ring will move upon a straight rod revolving about one of its ends on a horizontal plane, when, after the first percussive impulse, they are disturbed by no force but their mutual pressure against each other.

XVI. QUESTION 306, *by Mr.* W. WALLACE.

If the intersection of two straight lines which contain a given angle be on a line of any kind given by position, and one of them pass through a given point, what is the nature of the curve to which the other is a tangent?

XVII. QUESTION 307, *by Mr.* LOWRY.

Let AEC, BFC and EF be three straight lines given by position: From the given points A, B let straight lines be inflected to any point I in EF, meeting AC in P and BC in Q, and let P, Q be joined. Required the nature of the curve to which PQ is always a tangent?

XVIII. QUESTION 308, *by Mr.* W. WALLACE.

From two given points equally distant from the centre of a given circle draw straight lines to a point in the circumference, so that the sum of the fourth powers of these lines may be a given quantity.

XIX. QUESTION 309, *by* AMICUS.

A point and a straight line are given by position in the same vertical plane; a heavy body, urged by the force of gravity, has to descend from the given point in a circular arch, so as to arrive at the given line in the shortest time possible.—What is the length of the arch?

XX. PRIZE QUESTION 310, *by Mr.* W. WALLACE.

Find such integer values of x, y, z as ſhall render the three expressions $x^2 + axy + y^2$, $x^2 + a'xz + z^2$, $y^2 + a''yz + z^2$ ſquares, a, a', a'', being given numbers.

Solutions to these Questions must come to hand (Post Paid) *by June* 1, 1811.

THE

MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XIII.

I. QUESTION 311, *by Mr.* JOHN HAYNES, *Dublin.*

To divide a given square number n^2, into two such parts that the sum of their squares and the sum of their cubes may both be rational squares.

II. QUESTION 312, *by Mr.* P. PHELAN, *Carlow.*

Required the least integer value of a that will make $103\,a^2+1$, a rational square?

III. QUESTION 313, *by Mr.* GEO. HARVEY, *Jun. Plymouth.*

From a given right cone to cut such a parabolic section, that the solid generated by the revolution thereof, either about its axis, or its greatest ordinate may be the greatest possible.

IV. QUESTION 314, *by Mr.* JOSEPH WILLIAMS.

Emerson in his Increments has made use of a series to determine the integral of $\frac{z-n}{nz}\,\underset{.}{s}-\underset{.}{s}=\underset{n}{s}$: It is required to assign the integral in terms of s and $\underset{.}{s}$ without using a series?

V. QUESTION 315, *by* AMICUS.

If, from two given points, two straight lines be drawn to a point in a curve of any order, given by position, so that their rectangle may be the greatest or the least possible; these lines will have the same ratio as the secants of the angles which they make with the curve (or with the tangent) at the point of intersection. Required the demonstration?

VI. QUESTION 316, *by* G. V.

If from a given point straight lines be drawn to the extremities of any two conjugate diameters of an ellipse given by position, the sum of the squares of these lines is a given space. Required the demonstration?

VII. QUESTION 317, *by* G. V.

Let ABCD be a parallelogram, draw the diagonal BC, and draw DE perpendicular to BC; then, perpendiculars drawn to AB, AC at the points B and C shall intersect each other in the line DE. Required the demonstration?

VIII. QUESTION 318, *by* G. V.

If from one of the angles of a rectangle a perpendicular be drawn to its diagonal, and from their intersection lines be drawn perpendicular to the sides containing the opposite angle, then, putting P and p for these last perpendiculars, and D for the diagonal, $P^{\frac{2}{3}}+p^{\frac{2}{3}}=D^{\frac{2}{3}}$. Required the demonstration?

IX. QUESTION 319, *by Mr.* CUNLIFFE, *R M. College.*

Suppose a projectile delivered with a given velocity, what must be the angle of elevation that the length of the curve described in its flight above an horizontal plane may be the greatest possible, supposing no resistance from the air?

X. QUESTION 320, *by Mr.* CUNLIFFE.

In a given circle to inscribe a triangle having two of its sides in a given ratio, and the sum of the other side and perpendicular thereon from the opposite angle a maximum.

XI. QUESTION 321, *by Mr.* Cunliffe.

Suppose a piece of cast iron in the form of a cone, the diameter of the base being two, and the perpendicular height six feet. Suppose the cone suspended by the vertex, about which it is freely moveable: let two weights, of half a ton each, be fastened by cords to a ring in the circumference of the base, and let the cords pass freely over two pullies in the same horizontal line with the vertex, at the distance of four and five feet, on the same side of it. Required the position of the cone's axis when in equilibrio?

XII. QUESTION 322, *by Mr.* Lowry, *R. M. College.*

To find the nature of the curve which shall cut all the parabolas that can be described by a projectile, thrown from a given point, with a given velocity, so that the time of flight, from the given point to the curve, may be the same in each parabola.

XIII. QUESTION 323, *by Mr.* Lowry.

To find curves such, that any straight line being drawn through a given point to meet the curve in two points, the rectangle under it and a given line may be equal to the rectangle of the segments intercepted between the given point and the curve.

XIV. QUESTION 324, *by Mr.* Lowry.

With what radius must a circle be described from a given point as a centre, so that intersecting another circle given by position, the length of the arch intercepted by the given circle may be a maximum?

XV. QUESTION 325, *by Mr.* Lowry.

If two weights be suspended at the ends of two strings which pass over two fixed pullies, and the other ends of the strings be fastened to the extremities of a given beam or rod which is moveable about a fixed point, what will be the velocities of the weights when they move in a vertical direction, the weight of the beam or rod being considered in estimating the motion of the bodies?

XVI. QUESTION 326, *by Mr.* Lowry.

Suppose the vertex of a given cone to rest on an inclined plane while the axis is in a vertical direction; what will be the velocity of the centre of gravity of the cone when it is suffered to descend by the force of gravity, the vertex being at liberty to slide freely along the inclined plane without any obstruction whatever?

XVII. QUESTION 327, *by Mr.* W. Wallace, *R. M. College.*

Let there be three curves of such a nature, that if a tangent be drawn to one of them, a given portion of that tangent is intercepted between the other two; then, normals to the three curves at the points of intersection and contact shall pass through the same point. Required the demonstration?

XVIII. QUESTION 328, *by Mr.* W. Wallace.

Let A, B, two angles of a triangle ABC given in magnitude be upon lines of any kind. Draw AP, BP, normals to the lines at A and B, and join CP; this last line shall be a normal to the line that is the *locus* of C, the remaining angle of the triangle. Required the demonstration?

XIX. QUESTION 329, *by Mr.* W. Wallace.

Three curves being given by position, shew how a tangent may be drawn to one of them, so that a given portion of that tangent may be intercepted between the other two.

XX. PRIZE QUESTION 330, *by Mr.* Lowry.

A dog and a fox start at the same instant from two places M and N, and run at the uniform rate of m and n yards in a minute respectively: the dog proceeds in a straight direction MS, but the wily fox pursues such a course as prevents him from being seen by the dog, a fixed object on the level plain being always in a direct line between them. Required the nature of the curve described by the fox, when the distances of the object from M and N, and the angle NMS are given?

***Solutions to these Questions must come to hand* (post paid) *by June* 1, 1812.**

THE
MATHEMATICAL REPOSITORY.

MATHEMATICAL QUESTIONS.

To be answered in Number XIV.

I. QUESTION 331, *by Mr.* JOHN HYNES, *Dublin.*

To find any number of squares whose sum and product are equal.

II. QUESTION 332, *by Mr.* JOHN HYNES.

To find two fractions such, that the sum and sum of their squares shall both be rational squares; and either of them being added to the square of the other shall make the same square.

III. QUESTION 333, *by* JUNIUS.

Required the general value of x in the equation $x^2-23y^2=1$?

IV. QUESTION 334, *by Mr.* W. CALLOW.

There are two such quantities that the sum of their squares exceeds their sum by a, and that the sum of their fourth powers together with their sum, exceeds twice the sum of their cubes by b: It is required to find them without resolving any equation higher than a quadratic?

V. QUESTION 335, *by Mr.* CALLOW.

Having given $y^4+qy^2+ry+s=0$, and $y-(x+m)=0$: It is required to assign such a value to m, as will enable us to determine x and y, and thus to resolve a general biquadratic, supposing the solution of a general cubic to be known?

VI. QUESTION 336, *by* AIEYOYO.

Demonstrate the following Theorems:

1. If three circles be situated any how in a plane, and through the centres of every two a circle be described to touch the remaining one, the lines joining the centre of each of the circles with the point in which the circles passing through it meet each other intersect in the same point.

2. If there be three circles situated any how in a plane, and if through the centre of each, a circle be similarly described to touch the other two, then lines joining the centre of each of the circles with the intersection of those two which touch it, (taking always the corresponding intersections) will meet in the same point.

3. If through the centre of each of three circles situated in the same plane, tangents be drawn to the remaining two, either both to that part of the circumferences which is interior with regard to the three circles, or that which is exterior: then lines

drawn from the centre of each circle to the intersection of the two tangents drawn to it, will meet in the same point.

4. If three circles touch one another, the one of which is a mean proportional between the other two, and if tangents be drawn to every two of these meeting their diameters externally; then it is known that the three points of section are in the same straight line; and it is required to prove that the square of this line has a given ratio to $M^2 + N^2 + MN$; M and N representing the radii of the first and second terms of the proportion.

VII. QUESTION 337, *by Mr.* JOHN CAVILL, *Beighton.*

From two given points equally distant from the centre of a given circle to draw two lines to a point in the circumference so that their sum or difference may be equal to a given line.

VIII. QUESTION 338, *by* L. G.

Prove that the orthographic projection of any section of a spheroid upon the plane of its equator is a circle.

IX. QUESTION 339, *by Mr.* NOBLE, *R. M. College,*

Given the base, the difference of the angles at the base and rectangle of the sides, to construct the plane triangle.

X. QUESTION 340, *by Mr.* JOHN DAWES, *Birmingham.*

Given the base and vertical angle of a plane triangle to construct it, when the segment of one side, next the angle at the base, made by a line drawn parallel to the other side from the point in the base where the perpendicular from the vertical angle cuts it, is equal to a given line.

XI. QUESTION 341, *by* ANALYTICUS.

Required the fluent of $\dfrac{\dot{x}}{(3-xx)\sqrt[3]{(1-3xx)}}$?

XII. QUESTION 342, *by Mr.* CUNLIFFE.

Find the sum of n terms of the progression

$$1^2 \cdot 2^2 + 3^2 \cdot 4^2 + 5^2 \cdot 6^2 + 7^2 \cdot 8^2 \text{ \&c.?}$$

XIII. QUESTION 343, *by Mr.* CUNLIFFE.

A grey-hound spied a hare at the distance of half a mile, which he immediately pursued; now the hare fled away in a right line and the grey-hound made directly towards her: how far did the hare run before she was over-taken by the grey-hound, supposing the grey-hound's speed to that of the hare as 5 to 4, and the shortest distance from the grey-hound to the line of the hare's course, a quarter of a mile?

XIV. QUESTION 344, *by Mr.* R. J. DISHNEAH, *Trinity College, Cambridge.*

If the base of a cycloid revolve on the circumference of a circle whose radius is equal to the radius of the generating circle. Determine the nature of the curve traced out by the extremity of the ordinate of the cycloid belonging to the point of contact, and find the area (exterior to the circle) when every part of the base has been in contact with the circle,

XV. QUESTION 345, *by* AMICUS.

Divide a given oblique cone into two equal parts, so that the area of the section made by the cutting plane may be a minimum.

XVI. QUESTION 346, *by Mr.* P. BARLOW, *R.M. Academy.*

Admitting the masses, distances, and densities of the planets to be known, it is required to shew a direct method of ascertaining, whether any law has place between the powers or roots of those quantities, similar to that which obtains, with respect to their distances and times of periodic revolution.

XVII. QUESTION 347, *by* S.H. CHRISTIE, *R.M. Academy.*

Let $\mathrm{P}=\cos\frac{1}{n}a\,.\,\sin^3\frac{1}{n}a$, $\mathrm{P}'=\cos\frac{1}{n}a.\cos\frac{1}{n^2}a.n\sin^3\frac{1}{n^2}a$,

$$\mathrm{P}''=\cos\frac{1}{n}a\,.\,\cos\frac{1}{n^2}a\,.\,\cos\frac{1}{n^3}a\,.\,n^2\sin^3\frac{1}{n^3}a;\ \mathrm{P}'''=\&c.$$

$$\mathrm{Q}=\cos\frac{1}{n}a\,.\,\sin^5\frac{1}{n}a,\ \mathrm{Q}'=\cos\frac{1}{n}a.\cos\frac{1}{n^2}a.n\sin^5\frac{1}{n^2}a,$$

$$\mathrm{Q}''=\cos\frac{1}{n}a\,.\,\cos\frac{1}{n^2}a\,.\,\cos\frac{1}{n^3}a\,.\,n^2\sin^5\frac{1}{n^3}a;\ \mathrm{Q}''=\&c.$$

$$\mathrm{R}=\cos\frac{1}{n}a\,.\,\sin^7\frac{1}{n}a,\ \mathrm{R}'=\cos\frac{1}{n}a.\cos\frac{1}{n^2}a.n\sin^7\frac{1}{n^2}a,\ \&c.$$

Shew that

$$a=\frac{\sin a+\frac{n(n^2-4)}{2.3}(\mathrm{P}+\mathrm{P}'+\mathrm{P}''+\&c.)-\frac{n(n^2-4)(n^2-16)}{2.3.4.5}(\mathrm{Q}+\mathrm{Q}'+\mathrm{Q}''+\&c)+\&c}{\cos\frac{1}{n}a\,.\,\cos\frac{1}{n^2}a\,.\,\cos\frac{1}{n^3}a\,.\,\cos\frac{1}{n^4}a\,.\,\cos\frac{1}{n^5}a\,.\,\&c.}$$

when n is an even number, and that when n in is an odd number,

$$a=\sin a+\frac{n(n^2-1)}{2.3}(\sin^3\frac{1}{n}a+n\sin^3\frac{1}{n^2}a+n^2\sin^3\frac{1}{n^3}a+\&c.)$$

$$-\frac{n(n^2-1)(n^2-9)}{2.3.4.5}(\sin^5\frac{1}{n}a+n\sin^5\frac{1}{n^2}a+n^2\sin^5\frac{1}{n^3}a+\&c.)+\&c.$$

XVIII. QUESTION 348, *by Mr.* LOWRY, *R. M. College.*

Let CD be a diameter of an ellipse or hyperbola given by position, two points may be found such that, straight lines being drawn from them to any point in the curve, to meet CD in E and F, the segment EF shall be a mean proportional between CE and DF.

XIX. QUESTION 349, *by Mr.* LOWRY.

In what curve must a body be constrained to move upon the surface of an upright cylinder, so that when urged by the force of gravity, it may descend from one given point to another in the shortest time possible?

XX. PRIZE QUESTION 353, *by* G. V.

Suppose a conical mountain of a given figure to be situated in a plane. Find the length of the shortest road between two given points on the plane on opposite sides of the mountain.

Solutions to these Questions must come to hand (post-paid) *by June* 1, 1813.

NOTICES.

I. *Mathematical Works lately Published.*

The Philosophical Transactions of the Royal Society of London, for 1811. The mathematical papers contained in this volume are, 1. On the Expansion of any Functions of Multinomials. By Thomas Knight, Esq.—2. On the Rectification of the Hyperbola by means of two Ellipses, proving that method to be circuitous, and such as requires more calculation than is requisite by an appropriate Theorem; in which process a new theorem for the Rectification of that Curve is discovered, to which are added some farther observations on the Rectification of the Hyperbola; among which the great advantage of descending Series over ascending Series, in many cases, is clearly shewn; and several methods are given for computing the constant Quantity by which those Series differ from each other. By the Rev. John Hellins, B. D. F. R. S. being an appendix to his former Paper on the Rectification of the Hyperbola, inserted in the Philosophical Transactions for the year 1802.—3. On the Solar Eclipse which is said to have been predicted by Thales. By Francis Baily, Esq.—4. Astronomical Observations relating to the Construction of the Heavens arranged for the purpose of a critical examination, the result of which appears to throw some new light upon the Organization of the Celestial Bodies. By Dr. Herschel.

An Elementary Treatise on Astronomy. By Robert Woodhouse, A M. F.R.S. Fellow of Gonville and Caius College, Cambridge.

An Elementary Investigation of the Powers and Properties of Numbers, with their application to the theory of Arithmetic, the Indeterminate and Diophantine Analysis, the Division of the Circle, &c. By P. Barlow, of the Royal Military Academy, Woolwich.

A Treatise on the Resolution of the Higher Equations in Algebra. By W. Lea.

A Course of Mathematics designed for the use of the Officers and Cadets of the Royal Military College. By Isaac Dalby, Professor of Mathematics in the said College. Vol. I. Third Edition, corrected, with additions.

Outlines of Natural Philosophy; being Heads of Lectures delivered in the University of Edinburgh by John Playfair, Professor of Natural Philosophy, &c. Vol. I.

The History of the Royal Society. By Thomas Thomson, M.D. F.R.S.

II. *Mathematical Works in the Press.*

The Bija Ganita, or The Algebra of the Hindoos; Translated from the Persian. By Edward Strachey, Esq.

A new set of Mathematical Tables, containing Factors, Squares, Cubes, Square Roots, Cube Roots, Reciprocals, and Hyperbolic Logarithms of all Numbers from 1 to 10,000, the four latter columns true to 9 places of figures—An extensive Table of Fluents, some of the most useful Algebraical, Trigonometrical and Logarithmic Formulæ, and several other new and useful Tables, with an Introduction shewing their various uses and applications. By P. Barlow, of the Royal Military Academy, Woolwich.

THE
MATHEMATICAL REPOSITORY

MATHEMATICAL QUESTIONS.
To be answered in Number XV.

I. QUESTION 351, *by Mr.* JOHN BAINES, *Reading.*

To divide a given number into two parts, such that the sum of their squares, and the difference of their cubes may be squares?

II. QUESTION 352, *by* REDINTEGRATOR.

Given the three right lines bisecting the angles and terminating in the opposite sides to determine the triangle.

III. QUESTION 353, *by Mr* JAMES ADAMS, *Stone House, near Plymouth.*

To determine that point in the curve of a given ellipse which is at the greatest distance from the extremity of the conjugate axis.

IV. QUESTION 354, *by* ALIQUIS.

Given the two sides and the sum of the squares of the base and perpendicular of a plane triangle to construct it.

V. QUESTION 355, *by Mr.* S. JONES, *Liverpool.*

In a given circle to inscribe a triangle, such that the difference of its sides shall be in a given ratio to the difference of the segments of the base made by a perpendicular from the vertical angle, and the prism whose base is the triangle and altitude the said perpendicular a maximum.

VI. QUESTION 356, *by Mr.* CUNLIFFE.

To find two rational fractions, either of which being added to the fourth power of the other shall make the same sum.

VII. QUESTION 357, *by* QUIDAM.

Given the greatest and least slant sides of an oblique cone, to determine the diameter of the base, that its solid content may be the greatest possible.

VIII. QUESTION 358, *by* QUIDAM.

Find the sum of the infinite series $\frac{1}{3}-\frac{1.5}{3.7}+\frac{1.5.9}{3.7.11}-\frac{1.5.9.13}{3.7.11.15}+$ &c.

IX. QUESTION 359, *by Mr.* R. J. DISHNEAGH.

If any conic section revolve upon an equal conic section, the motion beginning from the vertex, it is required to shew how the locus of the vertex of the revolving conic section may be determined: and apply it to the case of a parabola.

X. QUESTION 360, *by the Rev. Mr.* L. EVANS, *of the Royal Military Academy, Woolwich.*

In fig.145. pl. 4, which represents a vertical section of a pendulum invented by Mr. *Adam Reid* of Woolwich, for which he has received a premium of the society of arts, Adelphi, AB is a steel rod, CD is a perforated cylinder of zinc supporting the bob E at its centre and resting on the regulating nut F turned by a screw at the lower end of the rod AB; so that as the steel rod contracts or expands, the zinc will be also affected, keeping the bob always at the same distance from the point of suspension. Now supposing the expansion of steel-rod to zinc to be in the proportion of 1147 to 2242 and that the pendulum is intended to vibrate seconds, what must be the length of the cylinder of zinc to effect the purpose of compensating for heat and cold, not taking into consideration the quantity of matter in the metals?

XI. QUESTION 361, *by* G. V.

Let CAP, CBQ be any two given semi-diameters of an ellipse; from any point V in the curve let VA be drawn parallel to a line given by position, to meet one of them in A; there is another line given by position to which if VB be drawn parallel, so as to meet the other diameter in B, the line joining A and B shall have a given ratio to CP the semi-diameter parallel to AB.

XII. QUESTION 362, *by* J. H.

To express the ratio (2π) of the circumference of a circle to radius in a formula involving only powers and roots of 2.

XIII QUESTION 363, *by* J. H.

If the series $\frac{x}{1^2} - \frac{x^2}{2^2} + \frac{x^3}{3^2} -$ &c. be denoted by ${}^2\text{L}\,(1+x)$, it is required to shew that the following equation holds,

$$ {}^2\text{L}\,(x) + {}^2\text{L}\left(\frac{1}{x}\right) = \frac{(\log.\, x)^2}{1.2}? $$

XIV. QUESTION 364, *by* J. H.

If a cylindrical vessel placed vertically and kept full of water, be bored in innumerable points, the issuing fluid will be bounded by the surface of a conical frustum. Required a proof.

XV. QUESTION 365, *by* J. H.

Let AM be the axis, MP an ordinate and PT the tangent of a common catenaria; AM and the tangent of MPT will be certain functions of MP, which we will denote by C (MP) and S (MP); then supposing the length of the portion of chain equal in weight to the tension at A, to be unity, the functions C and S will have the following properties,

1............ $\text{C}(\text{A})^2 - \text{S}(\text{A})^2 = 1$.
2............ $\text{S}(\text{A}+\text{B}) = \text{S}(\text{A}).\text{C}(\text{B}) + \text{C}(\text{A}).\text{S}(\text{B})$.
3............ $\text{C}(\text{A}+\text{B}) = \text{C}(\text{A}).\text{C}(\text{B}) + \text{S}(\text{A}).\text{S}(\text{B})$.
4............ $\text{C}(2\text{A}) = 2.\text{C}(\text{A})^2 - 1$.

XVI. QUESTION 366, *by* J. H.

If Peas be taken at random out of a bag, there is a greater probability of taking an odd than an even number: A proof is required.

XVII. QUESTION 367, *by* J. H.

Let $\epsilon = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} +$ &c. It is required to shew that the definite integral $\frac{1}{\pi}\int d\theta\,.\,\epsilon^{2\cos\theta}$ between the limits $\theta = 0$, $\theta = \pi = 4\left\{1 - \frac{1}{3} + \frac{1}{5} - \&c.\right\}$ is equal to the series $1^2 + \left(\frac{1}{1}\right)^2 + \left(\frac{1}{1.2}\right)^2 + \left(\frac{1}{1.2.3}\right)^2 +$ &c.

XVIII. QUESTION 368, *by* J. H.

Let S be the focus of a parabola $\text{P}_{(1)}\,\text{P}_{(2)}\ldots\ldots\text{P}_{(n)}$ whose vertex is A, axis AM, and latus rectum L. Draw any line $\text{SP}_{(1)}$ and make n angles $\text{P}_{(1)}\text{SP}_{(2)}$, $\text{P}_{(2)}\text{SP}_{(3)}\ldots\ldots\ldots\text{P}_{(n)}\text{SP}_{(1)}$ round S, all equal to each other. Draw SQ so that the angle $\text{MSQ} = n$ times the angle $\text{MSP}_{(1)}$: It is required to prove that

$$ \text{SP}_{(1)} \times \text{SP}_{(2)} \times \text{SP}_{(3)} \times \ldots\ldots\ldots \text{SP}_{(n)} = \text{L}^{n-1} \times \text{SQ}. $$

XIX. QUESTION 369, *by* J. H.

To find the form of equilibrium of an arch built from one planet to another, conceiving each particle animated with a force directed towards the sun, and varying inversely as the square of the distance from its centre.

XX. PRIZE QUESTION 370, *by Mr.* LOWRY.

If two given weights be fastened to the ends of a string, which passes over a fixed pulley, and the greater weight descend by the force of gravity, and draws up the other along a curve, which is in the same vertical plane as the pulley: required the nature of the curve when the time of ascent from one given point to another is the least possible.

The solutions to these questions must come to hand (post-paid) by June 1, 1814.

17. A given rectangular parallelogram is immersed vertically in a fluid with one side coincident with the surface. Divide it by a line parallel to the surface, into two parts that will be equally pressed.

TUESDAY AFTERNOON.—Mr. HUSTLER.

Third and Fourth Classes.

1. Prove that an harmonic mean is less than a geometric.

2. Find the amount of an annuity of 140*l*. payable quarterly, for three years, simple interest being allowed at 5 per cent.

3. In the common parabola, the normal is a mean proportional between the latus rectum and the focal distance. Required a proof.

4. Prove that the chord of 120° = tangent of 60°.

5. A body projected at an angle of 60° hits a mark at the distance of 300 feet upon an inclined plane whose elevation is 30°. Required the velocity of projection, the greatest height above the plane, and the time of flight.

6. Shew that the line joining the moon's cusps is perpendilar to the plane passing through the centres of the sun, earth, and moon.

7. If from the centre of an ellipse, with radius equal to the line joining the extremities of the axes, a circle be described; a body let fall from any point in its circumference towards the centre, will acquire at the point where it meets the ellipse the velocity which a body revolving in the ellipse about the centre would have at the same point.

8. The equations $x^3 - 3x^2 + 11x - 9 = 0$, and $x^3 - 5x^2 + 11x - 7 = 0$, have a common root. Find all the roots.

9. If a cylindrical vessel filled with a fluid and placed perpendicular to the horizon, empty itself through an orifice at the bottom, shew that the velocity of the decending surface will be uniformly retarded.

10. Prove according to NEWTON's 39th prop. that if the force vary as the distance, the velocity in a straight line is as the sine of a circular arc whose radius is the whole distance, and versed sine the space described.

11. Find the fluents of

$$\frac{x^3\dot{x}}{1 + x^2}, \text{ of } x^6\dot{x}\sqrt{(a^2 + x^2)}, \text{ and of } \frac{z^{\frac{1}{2}n-1}\dot{z}}{a^n + z^n}.$$

12. If the angle between the apsides in an orbit nearly circular be 60°, how does the force vary?

13. If a force varying as the distance tend to the centre of a globe, the times of oscillation in all arcs of the hypocycloid are equal.

TUESDAY EVENING.—Mr. HUSTLER.

1. Two bills, one for £a payable (b) months after date, the other for £c payable (d) months after date, are presented at a banker's, who advances £P for them. Required the rate of simple interest.

2. The excess of the sine above the versed sine is greater for 45° than for any other arc less than a quadrant.

3. Find the centre of gravity of a portion of a given paraboloid, cut off by a plane.

4. If the arc of a common cycloid in which a pendulum oscillates be divided into four equal arcs, the time through the first quarter = $\frac{1}{3}$ of the whole time of oscillation.

5. In the reciprocal spiral, the tangent of the angle made by the radius vector with the curve varies inversely as the distance. Required a proof.

6. With two dice, compare the chance of throwing the number 7 in one trial with the chance of throwing it twice in three trials.

7. If a body revolve in an ellipse, the force being in one focus, the angular velocity about the other focus is not *accurately* equal to the mean angular velocity, except at four points. Determine those points.

8. In the stereographic projection of the sphere, a great circle not passing through the pole is projected into a circle whose radius is the secant of inclination to the plane of the projection; and the distance of its centre from the centre of the sphere is the tangent of the same angle.

9. Sum the series $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4}$ &c. in inf.

10. The length and distance of a straight object placed before a concave spherical reflector being given, it is required to determine the axes of the image.

11. If several bodies be projected from different distances towards the centre, force varying as $\frac{1}{(\text{dist.})^2}$, with the velocities acquired in falling from infinity at those distances respectively; shew, according to NEWTON's 39th prop. that the times of falling into the centre are in the sesquiplicate ratio of the initial distances.

12. The roots of the equation $x^n - px^{n-1} + qx^{n-2} \ldots$

$- Qx + R = 0$, being a, b, c, &c. shew that $\frac{a^2}{b} + \frac{a^2}{c} +$ &c. $+ \frac{b^2}{a} + \frac{b^2}{c} +$ &c. $\frac{c^2}{a} + \frac{c^2}{b} +$ &c. $= (p^2 - 2q) \cdot \frac{Q}{R} - p$.

13. At a given place, to determine the day when a given star is due south at sun-rise.

14. A sphere acted upon by gravity is projected downwards in a medium with a velocity greater than the greatest acquirable velocity in the medium; determine the velocity after any space has been described, and the limit when the space is infinite.

15. If an ordinate parallel to the base of a common cycloid cut the curve in P and the generating circle in Q, and tangents PT, QT to the cycloid and circle be drawn, meeting in T, the locus of all the points T is the involute of the generating circle.

16. Find the fluents of $\frac{\dot{x}}{1 + x^n}$, (n) being an odd number, of $\frac{\dot{x}\sqrt{(a^2 - x^2)}}{x^6}$, and of $\frac{-\dot{x}}{\sqrt{(\text{hyp. log.} \frac{a}{x})}}$.

17. If the rhumb-line be always inclined to the meridians at 60°, its length from the equator to the pole = half a great circle of the sphere.

18. Determine the curve of aberration in a plane parallel to the ecliptic, if the orbit of the earth were a circle, and the sun in the circumference.

19. A cube is bisected diagonally by a plane, and one half being filled with a fluid is placed, vertex downwards, with the bisecting plane parallel to the horizon. Find the time of emptying through a small orifice made at the lowest part.

20. Supposing gravity to be constant and perpendicular to the horizon, and that the resistance of a medium varies as density of medium × (vel.^{2}) of the body; required the law of density, so that a body may describe a given circle. (NEWTON, vol. ii. prop. 10.)

21. A weight A (fig. 5. pl. A′.) attached to a string AB, being laid on a *horizontal* plane ABCD, the extremity B of the string is moved along a line BC which is at first perpendicular to BA, and the weight A traces out a curve AD on the plane. Shew that the surface of the solid made by the revolution of this curve indefinite in extent about BC = eight times its area.

22. Find the length of the tide-day at new and full moon, and shew that the tide-day is at all times greater than the solar-day.

23. The first term of an arithmetic series is (a), the last term (l), the common difference (d): also $\underset{1}{S}$, $\underset{2}{S}$, $\underset{3}{S}$ ..... $\underset{m-1}{S}$ are the sums of the first, second, third, $(m-1)^{\text{th}}$ powers of the terms: Shew that $(l+d)^m - a^m = m\,.\,d\,.\,\underset{m-1}{S} + m\,.\,\frac{m-1}{2}\,.\,d^2\,.\,\underset{m-2}{S} + m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3}\,.\,d^3\,.\,\underset{m-3}{S} + \&c.$

24. A great circle revolves about the axis of a sphere with an uniform angular velocity quadruple of that which a point setting off from the pole advances along it. Find the two surfaces into which the motion of the point divides the hemisphere.

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honors during the Examination for the Degree of B. A. in January, 1813.

BY THE TWO MODERATORS,

MONDAY, JANUARY 18, 1813.

MONDAY MORNING.—Mr. MACFARLAN.

First and Second Classes.

1. Sum the series, cos. A + cos. 2A + cos. 3 A.... + cos. nA where nA = whole circumference. Also the series $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5}$, &c. ad. inf.

2. Find the fluent of $\frac{\dot{x}}{\sqrt{(1+x^{\cdot})} - \sqrt{(1-x^{\cdot})}}$; and prove that the fluent of $x^x\dot{x}$ between the values of $x = 0$, and $x = 1$ is $\frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + $ &c.

3. AB (fig. 6. pl. A′) is a spherical reflecting arc; C the middle point; AT and BT tangents from the extremities, to which CP and CQ are drawn parallel: It is required in these lines to find two points, P and Q, so that all rays proceeding from P and incident at A or B may after reflection converge to Q.

4. Find the quantity of refraction by the circumpolar stars, (Boscovich's method) the refraction being supposed to vary as the tangent of the apparent zenith distance.

5. The angle contained by the two equal sides of an isosceles spherical triangle is greater than the angle contained by the chords of the same sides.

6. Find the value of the fraction $\frac{a^x - b^x}{x}$ when $x = 0$, and of $(1 - x)\tan.\frac{\pi x}{2}$ when $x = 1$, π being = semi-circumference of a circle rad. = 1.

7. If a straight line be drawn through the centre of gravity of a triangle to meet two sides and the third side produced; the rectangle under the segments of this line measured from the centre

of gravity on one side of it is equal to the sum of the rectangles under the same two segments and the segment on the other side of the centre of gravity.

8. Determine the position in which a lever of given length and uniform thickness will rest between two given inclined planes.

9. If the resistance vary as the velocity, and the force of gravity be constant, the times of describing all chords of a circle terminating in the extremity of a vertical diameter are equal.

10. In moving from the equator to the pole, the increase of a degree of latitude varies as the square of the sine of latitude,

11. A body revolves in a circle, the centre of force being in the periphery. Investigate the nature of the curve traced out by the extremity of the perpendicular upon the tangent; find its area and length, and the value of the greatest ordinate.

12. The density of a lever of given length varies as the n^{th} power of the distance from one extremity by which it is suspended. A given weight (P) attached to the other end, and acting perpendicularly by means of a pulley keeps the lever horizontal. The lever (when P is removed) would vibrate m times in t''. Required the weight of the lever, and the index (n).

13. A body, urged by a force varying inversely as the square of the distance, describes from rest a given straight line, while the line itself revolving uniformly performs one complete revolution. Required the area described.

14. The perimeter of an equilateral triangle inscribed in a circle is greater than the perimeter of any other isosceles triangle inscribed in the same circle.

MONDAY AFTERNOON.—MR. MACFARLAN.

Fifth and Sixth Classes.

1. Reduce $\frac{x^8 + x^6y^2 + x^2y + y^3}{x^4 - y^4}$ to its lowest terms; and prove the Rule for finding the greatest common measure of two quantities a and b.

2. Extract the square root of 315·271, and show generally that if there be n figures in the root there cannot be more than $2n$ nor less than $(2n - 1)$ figures in the number whose root is to be extracted.

3. Prove that the hypothenuse of a right-angled triangle is less than the sum of the two sides by the diameter of the inscribed circle.

4. Given the tangents of two arcs; find the tangents of their sum and difference.

5. When the force varies inversely as the square of the dist. the periodic time in ellipses vary as $\frac{(\text{axis major})^{\frac{3}{2}}}{\sqrt{(\text{abs. force})}}$.

6. Solve the following equations:

(1). $xz = y^2, \quad x + y + z = 21, \quad x^2 + y^2 + z^2 = 189.$

(2). $\sqrt{(5 + x)} + \sqrt{x} = \frac{15}{\sqrt{(5 + x)}}.$

(3). $x^2 + y^2 = 34, \quad x^2 - xy = 10.$

7. Prove that a geometric mean between two quantities is a mean proportional between an arithmetic and an harmonic mean between the same two quantities, and show of these three mean terms which is the greatest.

8. Extract the square root of $a^2 - b^2$ to four terms by the binomial theorem.

9. Find the fluxions of $(a + x) \times \sqrt{(a - x)}$; of ax and the fluent of $\frac{2a\dot{x}}{a^2 - x^2}$.

10. A body is projected from the bottom of a given inclined plane with a given velocity; find the direction when the range will be a maximum.

11. Find the focal length of a glass sphere.

12. The specific gravity of gold and silver being (a) and (b) and of their compound (c). Find the ratio of the quantities of the gold and silver in the mixture.

13. Given the latitude of the place and the sun's altitude at six o'clock. Find the time of the year, and give the proportions for solving the spherical triangle.

14. Construct the supplemental triangle, and prove its properties.

15. In the parabola the rectangle under the principal latus rectum and the abscissa is equal to the square of a semi-ordinate to the axis.

16. Prove the rule for the extraction of the square root of a binomial surd, and apply the expression to $7 - 2\sqrt{3}$.

17. In a triangle whose sides are (a) and (b), and included angle $\frac{4}{3}$ of a right angle, the square of the base $= \frac{a^3 \backsim b^3}{a \backsim b}$.

MONDAY AFTERNOON.—MR. JEPHSON.

Third and Fourth Classes.

1. Investigate the rule for extracting the square root of a binomial surd, and apply it to find the root of $2+2\sqrt{(1-a^2)}$.

2. The true zenith distance of the polar star when it first passes the meridian is $46^\circ\ 50'\ 40''\ 75'''$ and at the second passage is $50^\circ\ 25'\ 50''\ 30'''$. Required the latitude of the place.

3. If any number of circles be drawn through the given points A and B cutting a given circle, the lines which join the points of intersection shall all meet AB produced in the same point.

4. In a system of (n) equal pullies, each hanging by a separate string, and the strings parallel, having given P and W and the weight of one pulley, find (n) when there is an equilibrium.

5. If a vessel be filled with fluid, the pressure on any part : the weight of the fluid : : area of that part × the depth of its centre of gravity : solid content of the fluid.

6. If $p+1:1$ be the ratio of the tangents of two angles, and $m:n$ the ratio of their sines, shew that $p+1:1$ is always greater than $m:n$.

7. Prove that $\int \frac{\dot{x}}{\sqrt{(x^2+2ax)}} = \frac{1}{2}$ hyp. log. $\frac{\sqrt{x}+\sqrt{(x+2a)}}{\sqrt{2}}$,

and that $\int \frac{m\dot{z}}{a+bz^2} = \frac{m}{\sqrt{ab}}$. arc. tang. $= z\sqrt{\frac{b}{a}}$. (rad. 1).

8. Sum the series $2^2+4^2+6^2+8^2+$ &c. to (n) terms, and shew that the series $1^3+2^3+3^3+$ &c. to (n) terms, equals the square of the series $1+2+3+$ &c. to (n) terms.

9. Prove the following formulæ,

sin. $(x-z) =$ ultimately sin. $x - z$ cos. $x + \frac{z^2}{2}$. sin. x, (z) being diminished sine limite,

$$\frac{\text{cos. A} + \text{sin. A}}{\text{cos. A} - \text{sin. A}} = \text{tang. } 2\text{ A} + \text{sec. } 2\text{A},$$

and having given an arc A, find another arc B so that tang. B = sec. A — tang. A.

10. Force $\propto \frac{1}{(\text{dist.})^2}$ shew that the velocity at any point of

the descent $\propto$ tan. $\left(\frac{\theta}{2}\right)$. θ being the circular arc whose diameter is the first distance, and versed sine the space described.

11. The increment of a semi-circular area made by ordinates perpendicular to the diameter: the contempory increment of the corresponding sector : : versed sine of twice the arc : radius. Required proof, and find that point at which the difference between the sector and the area is a maximum,

12. Transform the equation $y^3 + 2py^2 - 33p^2y - 14p^3 = 0$ into one whose co-efficients are *numeral.*

13. Determine that point in an ellipse, force in focus, where the velocity is a harmonic mean between the greatest and least velocities.

14. Find the content of the greatest cone that can be cut out of a given paraboloid, the vertex of the cone being in the centre of the base of the paraboloid.

15. Find that point in P's orbit (11th section) at which the tangential ablatious force : the mean additious : : 3 : 2.

MONDAY EVENING.—Mr. JEPHSON.

1. Extract the square root of $4mn \pm 2 . (m^2 - n) . \sqrt{(-1)}$.

2. There are (p) arithmetical progressions each beginning from unity, the common differences are 1, 2, 3, ... p, shew that the sum of their nth terms $= \frac{(n-1) . p^2 + (n+1) . p}{2}$.

3. Force to S $\propto g . \frac{1}{(\text{dist.})^2}$ prove that the velocity acquired in descending down any space AC : that which would have been acquired at C if the force at A had continued constant : : the chord : the sine of a circular arc whose diameter is SA and versed sine AC.

4. If systems of logarithms be taken, whose bases increase in geometrical progression, shew that their moduli decrease in harmonical progression.

5. Given the sun's declination, and that the sun is due east when half the time between his rising and twelve o'clock is elapsed; find the latitude of the place.

6. There are two events A and B independent on each other: the probability of A's happening : probability of failing : : $p : y$; the probability of B's happening : probability of failing : : $r : s$. In ($2n$) trials what is the probability that they will happen alternately all along?

7. Trace and construct the curve whose equation is $y^2 = \frac{x^3 + bx^2}{c - x}$ and determine the acute angle at which it cuts the axis.

8. The concave surface of a cylinder filled with fluid is divided by horizontal sections into (n) annuli in such a manner that the pressure upon each annulus is equal to the pressure upon the base; given the radius of the cylinder, find its height, and also the breadth of the (p^{th}) annulus.

9. Find $\int \frac{\dot{x}}{x(a^2+x^2)^{\frac{5}{2}}}$; also having given $\int (a+cz^n)^m \, . \, z^{vn-1}\dot{z}$ find $\int (a + cz^n)^{m+p} \, . \, z^{vn-1}\dot{z}$; also find the values of y in the equation $y - \frac{\ddddot{y}}{\dot{z}^4} = 0$.

10. Sum $\frac{5}{1 \cdot 2 \cdot 1 \cdot 3} + \frac{9}{2 \cdot 3 \cdot 3 \cdot 5} + \frac{13}{3 \cdot 4 \cdot 5 \cdot 7} +$ &c. to (n) terms and in inf. and $\frac{1}{1 \cdot 3^2} + \frac{2}{3 \cdot 5^2} + \frac{3}{5 \cdot 7^2} +$ &c. in inf. and apply the method of increments to sum (n) terms of the series $1 + \frac{3}{1 \cdot 3} + \frac{3 \cdot 4}{2 \cdot 3^2} + \frac{4 \cdot 5}{3 \cdot 3^3} +$ &c.

11. Find the equation to a spiral in which the angle described by the radius vector $SP \propto \frac{1}{SP^n}$; and shew that the subtangent to any point P: the corresponding circular arc described with radius SP and beginning from the asymptote :: n : 1.

12. A revolving spheroid will retain its form if four times the primitive gravity at the equator : five times the centrifugal force of rotation :: semi-axis : the elevation of the equator above the inscribed sphere.

13. Two equal weights sustain each other by means of three tacks situated in the same vertical plane; prove that the *vertical* pressures are together equal to the sum of the weights.

14. An inverted paraboloid is supplied with water at a given rate; given its dimensions, and the area of the orifice which is in the vertex: It is required to find the highest point to which the water will rise, and also the time.

15. Parallel rays are incident upon a semi-cycloid in a direction

perpendicular to the base, find the caustic, and shew that the density $\propto$ tang. θ + 2 tang. 2θ (θ being that arc of the generating circle of the cycloid which corresponds to the point of incidence.)

16. If (v) = the true anomaly, u = the eccentric, and $e : 1$, the ratio between the eccentricity of the orbit, and its semi-axis major, then $\cos. v = \frac{e + \cos. u}{1 + e . \cos. u}$. Required proof.

17. A logarithmic curve being described, construct for its sub-tangent.

18. P and Q are placed at the ends of a lever, P hangs by a thread; given P and Q, it is required to find where the fulcrum must be placed so that the tension of the thread may be a maximum.

19. A body oscillating in a medium whose resistance $\propto (\text{vel.})^2$, construct for the resistance at each point. (Newt. prop. 29, vol. ii.)

20. By means of the compasses alone, it is required to inscribe in a square an equilateral triangle having one angle in an angle of the square.

21. The corner of a rectangular piece of paper is doubled down, so that the triangle shall always be of a given area, prove that the vertex of the triangle will trace out a lemniscata, whose area equals the area of the triangle and which may be described by a force placed in its *knot* varying as $\frac{1}{D^7}$.

22. A quadrant is stretched out into a straight line, and upon it as an axis ordinates are drawn which are equal to the versed sine of twice the intercepted arc; find the whole area of the curve so traced out, and determine the point of contrary flexure.

23. The nodes being in quadratures, prove that the mean decrement of their motion arising from the acceleration of the areas is equal to $\frac{1}{4}$th of the decrement in syzygy.

24. If a perfectly flexible and uniform chain of a given weight coincide with the convex surface of a vertical quadrant having one radius horizontal, find the velocity acquired in its descent, and the tension at a given point in any given position of the chain.

TUESDAY MORNING.—Mr. JEPHSON.

First and Second Classes.

1. Are the quantities $\sqrt[4]{-a} \times \sqrt[4]{-b}$ and $\sqrt[6]{-a} \times \sqrt[6]{-b}$ possible or impossible?

2. Given that a parabolic area : its circumscribing parallelogram always : : $m : n$; it is required to find the ratio between the solids generated by these areas revolving round their common axis, and to apply it to the case of the common parabola.

3. Shew that at a point of contrary flexure the fluxion of $\frac{\dot{x}}{\dot{y}} = 0$.

4. The inscribed sphere is taken away from an oblate spheroid of small eccentricity; find that annulus parallel to the plane of the equator which attracts a corpuscle at the pole with a maximum force.

5. If $(-a)$ be a root of the equation

$x^n + px^{n-1} \mp px^{n-2} \pm$, &c..... $\pm W = 0$, shew

that $x + a$ is a divisor of the *expression*

$x^n + px^{n-1} \mp px^{n-2} \pm$, &c..... $\pm W$.

6. The latitudes of two places on the same meridian are observed, and perpendiculars to their horizons are drawn meeting within the earth: (not supposed to be a sphere); shew that the angle at which the perpendiculars will meet is equal to the difference of the observed latitudes.

7. Find the content of the solid generated by the revolution of a cycloid round a tangent parallel to the base.

8. Explain the Nonius, and shew that the instrument is rendered more *sensible* by increasing the number of the divisions.

9. A bag contains three red balls and two white ones; what is the probability of taking out a white ball in two trials?

10. Give a geometrical construction for finding the resistance upon any curve, and apply it to determine the ratio between the resistance upon a catenary moving in the direction of its greatest ordinate and the resistance upon its axis.

11. The circumference of a semi-circle being considered as an abscissa, and ordinates drawn from its convex side in a direction perpendicular to the diameter and varying as the n^{th} power of the intercepted arc, it is required to draw a tangent to

the quadratrix thus traced out, and to shew that in the case of the cycloid, the tangent is parallel to the corresponding chord of the generating circle.

12. A perfectly hard body falls down an inclined plane AB, and is inflected along another inclined plane BC; now AD, DE, EF are respectively perpendicular to CB produced, to AB and to the horizontal line CF; shew that the velocity acquired at C is equal to that which would be acquired by falling down EF.

13. Every recurring series whose scale of relation is $f - g + e$, may be resolved into three geometrical series, whose common ratios are the roots of the equation $x^3 - fx^2 + gx - e = 0$.

14. P hanging freely raises W up an inclined plane by means of a thread not parallel to the plane; find the tension of the string.

15. If (t) = the time of a comet's passage between its nodes, (q) = one year, $\pi = 3{\cdot}14159$, then will the line of nodes = $\sqrt[3]{\left(\frac{18\pi^2 t^2}{q^2}\right)}$, the earth's mean distance being (1). Required proof.

TUESDAY AFTERNOON.—Mr. JÉPHSON.

Fifth and Sixth Classes.

1. Shew that sin. $(-A) = -$ (sin. A), and that cos. $(-A)$ = cos. A. Is sec. $(-A) = +$ or $-$ sec. A?

2. Solve the following equations:

(1). $a + x + \sqrt{(2ax + x^2)} = b$.

(2). $\frac{y}{x} - \frac{x}{x+y} = \frac{x^2 - y^2}{y}$, $\frac{x}{y} - \frac{x+y}{x} = \frac{y}{x}$.

3. The earth a sphere, and its radius 4000 miles, find what distance may be seen by a person 9 feet high.

4. Divide $x^3 - px^2 + qx - r$ by $x - a$.

5. If two equal forces sustain each other by means of a string passing over a tack, shew that either of the forces : pressure upon the tack : : radius : 2 cosine of half the angle at which the forces act.

6. If $a : b :: c : d$, then will $a \pm mb : b :: c \pm md : d$.

7. Prove that the radius of a circle which bisects the chord will bisect the arc.

8. Shew that the versed sines of the same angle in different circles are to each other as the radii of the circles.

9. If A : B :: C : D, then will log. D = log. B + log. C — log. A.

10. An inverted paraboloid is filled with a fluid; find that horizontal section which sustains the greatest pressure.

11. Explain the magic lantern.

12. Shew the use of logarithms in finding the value of $\frac{A^2 \sqrt{(B^2 - C^2)}}{C \sqrt[5]{(D^3 . E . F)}}$.

13. Find all the combinations which can be made out of the letters of the word *Baccalaureus.*

14. A certain velocity (v) is communicated to each of two perfectly hard bodies at the instant of their impinging on each other. Shew that the common velocity after impact equal $a \pm$ what would have been the common velocity if (a) had not been communicated.

15. Find the 28th term of the series 13, $12\frac{2}{3}$, $12\frac{1}{3}$, &c. and sum the series 2, $-\frac{1}{3} + \frac{1}{18}, -\frac{1}{108}, +$ &c. in inf.

16. Take the fluxion of $(x^m + bx^n)^p$ and find the $\int \frac{\dot{x}}{a - mx}$, and $\int \frac{z^{\frac{n}{2}-1} \dot{z}}{a^n + z^n}$.

17. Prove that in all curves the centripetal force : the centrifugal :: SP : $\frac{1}{2} . \frac{QT^2}{QR}$.

18. Force to S $\propto$ dist.; shew that the velocity acquired in descending through any space AC : that which would have been acquired at C if the force at A had continued constant :: sine : the chord of a circular arc whose radius is SA, and versed sine AC.

TUESDAY AFTERNOON.—Mr. MACFARLAN.

Third and Fourth Classes.

1. Solve the equation $x^3 - 11x^2 + 36x - 36 = 0$, the roots being in harmonical progression.

2. If (n), a prime number, be the index of a binomial, every term of the expanded binomial, except the first and last, is divisible by (n).

3. If two bodies be acted upon by constant moving forces in the proportion of 5 : 4, describe spaces from rest in the proportion of 4 : 5, and acquire velocities in the proportion 5 : 6. Required the ratio of the quantities of matter.

4. In a given circle, the plane of which is vertical, to draw a diameter, which shall be described by a heavy body in any given time, not less than the time in which the vertical diameter is described.

5. The least angle, which can be made with the horizon by any great circle passing through the place of a star at any given time, is measured by the star's altitude.

6. The periodic times in all ellipses round the same centre are equal.

7. Find the effects of precession in right ascension and declination; and show when the effect in right ascension vanishes.

8. Find the fluent of $\dfrac{x^3\dot{x}}{\sqrt{(2ax - x^2)}}$ and of $\dot{z} \times (\sin. z)^3$.

9. Sum the following series:

(1). $1^2 + 3^2 + 5^2 + 7^2$ to 12 terms.

(2). $1 . x + 2x^2 + 3x^3 + 4x^4 \ldots\ldots + nx^n$.

(3). $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2}$, &c. ad. inf.

10. The density of the sun's rays formed by a spherical reflector : density of his rays formed by a glass sphere of the same radius :: 9 : 1. Required proof.

11. Compare, after Newton's method, the resistance upon the solid generated by the revolution of a cycloid round the base, moving in the direction of the base, with the resistance upon the circumscribing cylinder.

12. Having given the angle of a plane triangle, the side op-

posite to it and the sum of the other two sides. Required the sides.

13. Let a sphere descend by its gravity in a fluid, whose specific gravity is to that of the sphere as 1 to n. Find the greatest velocity it can acquire on supposition that the resistance varies as the square of the velocity.

14. Extract the square root of 2 by a continued fraction.

TUESDAY EVENING.—Mr. MACFARLAN.

1. If the national debt be A£. and P£. be annually invested at compound interest as a sinking fund, in how many years will the debt be discharged, the interest of A not being considered?

2. Find the sun's place in the ecliptic, when the aberration of a given star in declination vanishes; and shew that the aberration in right ascension is not necessarily a maximum when the aberration in declination $= 0$.

3. Find, after Newton's manner, the law of force whereby a body may describe a semi-ellipse, the direction of the force being perpendicular to the axis major.

4. When the force varies inversely as the fourth power of the distance, and a body is projected from an apse with the velocity acquired in falling from an infinite distance, to define the orbit.

5. The curve AVR (fig, 7, pl. A′.) and the semi-circle APB have the same abscissa; the ordinate MV is equal to the tangent of half the arc AP: Prove that the area AMV is equal to twice the circular segment AP; and find the point of contrary flexure.

6. Sum the series $\frac{5}{1 \cdot 2 \cdot 3} + \frac{6}{2 \cdot 3 \cdot 4} + \frac{7}{3 \cdot 4 \cdot 5}$ to n terms, and $\frac{6^2}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{7^2}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{8^2}{3 \cdot 4 \cdot 5 \cdot 6}$ to n terms and ad infinitum.

7. Find the chance of throwing three aces exactly in five throws with a single die.

8. Find the greatest of all triangles having the same vertical angle and equal distances between that angle and the bisection of the opposite side.

9. In orbits of little eccentricity, the greatest equation of the centre is equal to twice the eccentricity.

10. Integrate the following fluxional equations:

$$\frac{\dot{y}}{y}-\frac{\dot{x}}{x}=\frac{x^m\dot{x}}{ay\sqrt{n}},\ x\dot{x}+a\dot{y}=b\sqrt{(\dot{x}^2+\dot{y}^2)},\ y\dot{x}-x\dot{y}=\dot{x}\sqrt{(x^2+y^2)}.$$

11. Find the fluent of $\frac{(1+\sqrt{x}-\sqrt[3]{x^2})\dot{x}}{1+\sqrt[3]{x}}$ and of $\frac{\dot{x}}{(1+x^2)^n}$ and when $n=3$ shew that the value of the latter fluent between the value of $x=0$ and $x=1$, is $\frac{1}{4}+\frac{3A}{8}$, A being an arc of 45° to radius 1.

12. If any number of projectiles be thrown from the same elevated point, and with the same velocities in an horizontal direction; the locus of the points in which they will strike a given inclined plane will be a conical frustum.

13. If a cylindrical vessel, placed vertically, and kept full of water, be bored in innumerable points; the issuing fluid will be bounded by the surface of a conical frustum.

14. Find the equation between the abscissa and ordinate of the catenary.

15. A body (P) draws a lighter body (W) over a fixed pulley. A small oscillating motion is given to W at the commencement of P's action. Find the number of oscillations before W reaches the pulley: Show that this number is independent of the string's length; and that however great P is, $\frac{1}{\sqrt{2}}$ × oscillation at least will be performed in the time specified.

16. Two rods, the one 6, the other 8 feet high, are placed on a given day perpendicularly to the horizon, at the distance of 20 feet from each other. During the forenoon the extremity of the shadow of the first rod falls at the base of the second. In the afternoon the extremity of the shadow of the second falls at the base of the first. Required the latitude of the place, and the azimuth of one rod seen from the other.

17. If a body be projected from an apse with a given velocity, the force acting perpendicularly to a given plane, and varying in some inverse ratio of the distance from it, investigate the fluxional equation to the curve which will be described, and apply it to the case where the force is constant.

18. Supposing the sun to move uniformly on in a right line with a given velocity, and the earth revolving round him, to preserve always the same distance; it is required to define the earth's path in fixed space.

19. A body describes the quadrant of a circle touching a vertical line at its highest point, being urged by a force per-

pendicular to the horizon. Required the law of force which will make it recede uniformly from the horizontal radius, and the time elapsed and the velocity acquired at any point of the descent.

20. If any number, a multiple of 11, and a number consisting of the same digits in an inverted order, be each divided by 11, the sum of the digits in the two quotients are equal. Required a proof.

21. If n be a prime number, the product of $1 \times 2 \times 3 \times 4 \ldots \times (n - 1)$ when increased by unity is divisible by n.

22. If the co-efficients of each term of the expanded binomial $(a - b)^n$ taken in order, be multiplied by the terms of the progression $1^m, 2^m, 3^m$, &c. taken in order, the result is equal to nothing, n and m being integers and n greater than m. Required a proof.

23. Let α, β, γ, &c. be the roots of the equation $x^n - nax^{n-2} + n \cdot \frac{(n-3)}{2} a^2 x^{n-4} -$ &c. $= -b$; then $\alpha^{2r} + \beta^{2r} + \gamma^{2r} +$ &c. $= \frac{2r \cdot (2r-1) \cdot (2r-2) \ldots (r+1)}{1 \cdot 2 \cdot 3 \ldots\ldots\ldots\ldots r} \times na^r$ ($2r$ less than n.) Required a proof.

24. A body, whose weight is W falls down the length of an inclined plane, which has the power of moving freely along an horizontal plane, on which it stands. Given the weight of the prismatic figure composing the plane; it is required to find the path of the body W, the time of describing it, and the last acquired velocity of the moveable plane along the horizontal plane.

THE

MATHEMATICAL REPOSITORY.

VOL. III. PART I.

MATHEMATICAL QUESTIONS.

ARTICLE I.

Solutions to Questions proposed in Number VIII.

I. QUESTION 241, *by* HELOZAM SATBY.

A person standing 35 feet from the foot of an obelisk, threw a stone just over the top of it, which fell 41 feet from the obelisk on the other side, and the time of the stone's flight was observed to be 3 seconds. Now, supposing the stone was discharged six feet above the foot of the thrower, and the ground horizontal; it is required to determine the height of the obelisk, neglecting the air's resistance?

FIRST SOLUTION, *by Mr.* JOHN CAVILL, *Beighton.*

Let Dv (fig. 1. pl. 1.) represent the obelisk perpendicular to the horizontal line AB; P the point at which the stone is delivered; PvB the parabola described by the stone in its flight, and PE a tangent to the curve at P. Let BE be perpendicular to AB meeting the tangent in E, and suppose Pn parallel to AB meeting Dv in m, the curve in p, and BE in n; and let PA be perpendicular to AB; also let Dm be produced to meet PE in F. The whole time of the stone's flight, by the question, is three seconds, which is well known to be equal to the time in which a heavy body, by the force of gravity, would freely descend from rest at E, along the right line EB. Therefore by the laws of falling bodies $\mathrm{BE} = 9 \times 16\tfrac{1}{12} = 144\tfrac{3}{4}$ feet. Then by the property of the curve

$$\mathrm{P}n^2 : \mathrm{P}m^2 :: \mathrm{BE} : \mathrm{F}v = (\mathrm{P}m^2 \div \mathrm{P}n^2) \times \mathrm{BE} = 30\cdot69923.$$

Third

And by reason of the parallels, BE, DF

$Pn : Pm :: En : mF = Pm \times En \div Pn = 63{\cdot}89802$.

Then we shall obviously have $mv = mF - Fv = 33{\cdot}19879$, and $Dv = Dm + mv = 39{\cdot}19879$ feet the required height of the obelisk.

SECOND SOLUTION, *by Mr.* THOS. BAZLEY, *Bolton.*

Let AB (fig. 1. pl. 1.) be the horizontal plane, DLv the obelisk perpendicular thereto, and AP (parallel to Dv) = six feet: let PFE be the line of projection meeting the vertical line BE in E and the obelisk produced in F; draw PLB meeting Dv in L, and let AP = 6 feet = a, AD = 35 feet = b, DB = 41 feet = d, Dv = x, $t = 3$ seconds = time of flight, and $g = 16\frac{1}{12}$ feet.

Then by the principles of constant forces BE is $= gt^2$, and by similar triangles $b + d : b :: gt^2 : \text{LF} = bgt^2 \div (b + d)$; and by uniform motion $b + d : b :: t : bt \div (b + d) =$ time in PF = time in Fv: consequently $Fv = g\,(bt \div (b + d))^2$.

Again, $b + d : d :: a : \text{DL} = ad \div (b + d)$; whence

$$Dv = \frac{ad}{b+d} + \frac{bgt^2}{b+d} - g\left(\frac{bt}{b+d}\right)^2 = x = 39{\cdot}1871 \text{ feet.}$$

Solutions were also received from Messrs. Adams, Crane, Filmatt, Grisenthwaite, Harvey, Kay, Marratt, Swale, Toplis, *and* Watts.

II. QUESTION 242, *by Mr.* GERMAN BUXTON.

To find two numbers such, that if either of them be subtracted from the square of the other, the remainder shall be a square number?

SOLUTION, *by Mr.* WM. MARRAT, *Boston.*

Let x and y denote the two numbers.

Then by the question $x^2 - y$ and $y^2 - x$ must both be squares.

Put $x^2 - y = (r - x)^2 = r^2 - 2rx + x^2$ which gives $x = (r^2 + y) \div 2r$;

theref. $y^2 - x = y^2 - \frac{r^2+y}{2r} =$ a square $= (s - y)^2 = s^2 - 2sy + y^2$;

whence $y = \frac{2rs^2+r^2}{4rs-1}$ and $x = \frac{r^2+y}{2r} = \frac{2r^2s+s^2}{4rs-1}$ which are general expressions for the two numbers where r and s may be expounded by any unequal rational numbers at pleasure.

Ex. 1. Take $r = 2$ and $s = 1$; then $x = \frac{9}{7}$ and $y = \frac{8}{7}$.

Ex, 2.

Ex. 2. Take $r = 3$ and $s = 1$; then $x = \frac{8}{11}$ and $y = \frac{16}{11}$.

Solutions were also received from Messrs. Adams, Cavill, Crane, Filmatt, Grisenthwaite, Harvey, Smith, Swale, Toplis, *and* Watts.

III. QUESTION 243, *by* HYPATIA.

An arc of a given circle, being given in magnitude, draw a tangent to it, so that the part of the tangent intercepted between lines drawn from the centre through the extremities of the given arc, may be given in magnitude?

FIRST SOLUTION, *by Mr.* J. WALLACE, *Teacher of the Mathematics in Edinburgh.*

Let AB (fig. 2. pl. 1.) be the given arc, C the centre of the circle, and DE a straight line to which the tangent is to be equal. Draw AF perpendicular to CA; make AF equal to DE, and upon it describe the semi-circle AGF; from G, the point of intersection of this semi-circle with the radius CB produced, draw GH perpendicular to CG and let it meet AF in H; draw CH intersecting the arc AB in K; then through the point K draw the tangent LKM; and the part LM of this tangent intercepted between CL and CM shall be equal to DE.

For join AG, GF: then since the angle GAC is equal to GFH, and CGA to FGH; the triangles CGA, GHF are similar; hence CG : CA or CK :: GH : HF. But from the similar triangles CGH, CKM, we have CG : CK :: GH : KM; therefore by equality of ratios GH : HF :: GH : KM; hence KM is equal to HF. But it is obvious that KL is likewise equal to HA; consequently the whole LM is equal to the whole AF; that is, to the given line DE.

SECOND SOLUTION, *by* TYROTATOS.

Let AB (fig. 3. pl. 1.) be the given arch, and C the centre of the given circle. Make AM perpendicular to AC and equal to the given length of the tangent; draw MI parallel to AC meeting CB produced in I. On the diameter IC describe a semicircle cutting MA in L; draw CL to meet the circle in D, and through D draw the tangent EF, which is that required.

Draw IL to meet CA produced in G; then the angle CLI (in the semi-circle) being a right angle, the line IG is parallel to the tangent EF; and the triangles ALG, LIM being similar, we have, AL : LM :: GL : LI :: ED : FD.

But AL is = to ED, therefore LM is equal to FD, and consequently EF is = AM = the given length.

Third

THIRD SOLUTION, *by Mr.* GERMAN BUXTON, *London.*

Let AB (fig. 4. pl. 1.) be the given arc, and *pq* the given right line; and draw the radii CA, CB. Bisect the given line *pq* by the perpendicular *mo*, and draw *po* making the angle $pom =$ the angle ACB. Upon AC take $An = mo$, and draw *n*O at right angles to AC; from the point C to *n*O apply $CO = po$. With the centre O and radius OC, describe a circle cutting the right line PQ, perpendicular to AC, in P and Q, and join CP, CQ. On CA produced take $CE = CP$ and from E draw EDF touching the given arc in D, and meeting the radius CB in F, and the thing is done.

DEM. Draw the radius CD to the point of contact D, also draw OM perpendicular to PQ; then $MO = An = mo$; and $OP = OQ = op$, wherefore the right-angled triangles POM, *pom*, are equal in all respects, and therefore $PM = pm$ and $\angle POM = \frac{1}{2}\angle POQ = \angle PCQ = \angle pom = \angle ACB$, by construction and Euc. 20. III. Again in the right-angled triangles CAP, CDE, $CE = CP$ and $CD = CA$, therefore $ED = PA$ and $\angle ECD = \angle PCA$, and from the equal angles ACB (ECF), PCQ taking the equal angles ECD, PCA there will remain $\angle DCF = \angle ACQ$, and therefore $DF = AQ$; consequently $EF = ED + DF = PA + AQ = PQ = pq$, the given line.

Solutions were also received from Messrs. Adams, Butterworth, Cantab. Caiensis, Cavill, Grisenthwaite, Filmatt, Harvey, Kay, Marratt, Reeves, Swale, Toplis, *and* Watts.

IV. QUESTION 244, *by* SCOTUS.

In the isosceles triangle ABC, having $AB = BC$, if the point D be taken in AB, and DC joined. The solid under AC^2 and DB, is equal to the difference of the solids under AB, DC^2 and AD^2. Required the demonstration?

FIRST SOLUTION, *by Mr.* J. T. M'DONELD.

DEM. Draw DE (fig. 5. pl. 1.) perpendicular to AC; and DF, DG parallel to BC, AC.

Then by similar triangles $AC : AB :: FC\ (DG) : DB$; hence $AC \cdot DB = AB \cdot FC$, and $AC^2 \cdot DB = AB \cdot AC \cdot FC$.

Now $AC \cdot FC = FC^2 + 2EF \cdot FC = CD^2 - DF^2 = CD^2 - AD^2$, by Euc. II. 12. Theref. $AC^2 \cdot DB = (CD^2 - AD^2) \cdot AB$.

SECOND

SECOND SOLUTION, *by Mr.* J. ADAMS, *Stonehouse, near Plymouth.*

On AB (fig. 6. pl. 1.) demit the perpendicular CE; then per Euc. VIII. 6. $2AB \times AE = AC^2$, and Euc. II. 13. $DC^2 + 2AD \times DE = AD^2 + AC^2$.

Therefore
$DC^2 - AD^2 = AC^2 - 2ADE = 2BAE - 2DAE = 2BD.AE$.

Hence $(DC^2 - AD^2) \times AB = 2BD \times AE \times AB = AC^2 \times BD$.

THIRD SOLUTION, *by Mr.* ROBERT SMITH, *near Manchester.*

Draw DG (fig. 5. pl. 1.) parallel to AC meeting BC in G and join AG. Then AG and CD are manifestly equal to each other. Also ∠ BAC = BCA = supp. DGC. Therefore a circle will pass through the points A, D, G, C. Then by a well known property of a trapezium inscribed in a circle

$$AC \times DG + AD \times CG = AG \times CD, \text{ that is}$$

$$AC \times DG + AD^2 = DC^2, \text{ or } AC \times DG = DC^2 - AD^2,$$

$$\text{and } AB \times AC \times DG = AB \times (DC^2 - AD^2).$$

Again by reason of the parallels DG, AC,
$DG : AC :: BD : AB$, whence $DG \times AB = BD \times AC$: wherefore $AB \times AC \times DG = BD \times AC^2 = AB \times (DC^2 - AD^2)$.

Solutions were also received from Messrs. Butterworth, Cavill, Filmatt, Harvey, Kay, Marratt, Reeves, Swale, Toplis, *and* Watts.

V. QUESTION 245, *by* X.

Required the fourth root of $-a$?

FIRST SOLUTION, *by Mr.* WM. CRANE, *Boston.*

By squaring the binomial $\sqrt{a} \pm \sqrt{-a}$, we get

$(\sqrt{a} \pm \sqrt{-a})^2 = a \pm 2a^{\frac{1}{2}}\sqrt{-a} - a = \pm 2a^{\frac{1}{2}}\sqrt{-a}$;
hence

$$\pm\sqrt{-a} = \frac{(\sqrt{a} \pm \sqrt{-a})^2}{2a^{\frac{1}{2}}}$$

and extracting the square root;

$$\sqrt[4]{-a} = \pm\frac{(\sqrt{a} + \sqrt{-a})}{\sqrt{2} \times a^{\frac{1}{4}}} = \pm a^{\frac{1}{4}} \times \left(\frac{1 \pm \sqrt{-1}}{\sqrt{2}}\right), \text{ the answer}$$

required.

SECOND

SECOND SOLUTION, *by Mr.* W. GRISENTHWAITE.

Let $a^{\frac{1}{4}}y$ denote the required root; then $ay^4 = -a$, or $ay^4 + a = 0$; therefore $y^4 + 1 = 0$. Put $(y^2 + 1) \div 2y = v$, or $y^2 + 1 = 2vy$, then $(y^2 + 1)^2 = y^4 + 2y^2 + 1 = 4v^2y^2$; $y^4 + 1 = 4v^2y^2 - 2y^2 = y^2(4v^2 - 2) = 0$, or $4v^2 - 2 = 0$; whence $v = \pm\sqrt{\tfrac{1}{2}}$. And from the equation $y^2 + 1 = 2vy$,

$$y = v \pm \sqrt{(v^2 - 1)} = \pm\sqrt{\tfrac{1}{2}} \pm \sqrt{\frac{-1}{2}} = \pm\sqrt{\tfrac{1}{2}} \times (1 \pm \sqrt{-1});$$

consequently $a^{\frac{1}{4}}y = \pm a^{\frac{1}{4}}\sqrt{\tfrac{1}{2}} \times (1 \pm \sqrt{-1})$.

THIRD SOLUTION, *by Mr.* WM. MARRAT, *Boston.*

Let x be the root required and $a = b^4$; then $x^4 = -a = -b^4$, or $x^4 + b^4 = 0$. Assume $x^4 + b^4 = (x^2 - cx + b^2)(x^2 + cx + b^2)$; then by multiplication

$x^4 + b^4 = x^4 + 2b^2x^2 - c^2x^2 + b^4 = 0$, or $2b^2x^2 - c^2x^2 = 0$; dividing by x^2 and extracting the root we get $c = b\sqrt{2}$.

Hence $x^4 + b^4 = (x^2 - bx\sqrt{2} + b^2)(x^2 + bx\sqrt{2} + b^2) = 0$; consequently $x^2 - bx\sqrt{2} + b^2 = 0$, and $x^2 + bx\sqrt{2} + b^2 = 0$; and by resolving these two quadratics we get

$$x = \frac{b}{\sqrt{2}} \pm \frac{b}{\sqrt{2}}\sqrt{-1} = \frac{a^{\frac{1}{4}}}{\sqrt{2}}(1 \pm \sqrt{-1}),$$

$$\text{and } x = -\frac{b}{\sqrt{2}} \pm \frac{b}{\sqrt{2}}\sqrt{-1} = \frac{a^{\frac{1}{4}}}{\sqrt{2}}(-1 \pm \sqrt{-1}).$$

Solutions were also received from Messrs. Filmatt, Harvey, Toplis, *and* Watts.

VI. QUESTION 246, *by Mr.* BAZLEY.

Suppose a spring to be compressed through a space of 6 inches, and that a weight or pressure of 64lbs. retains it in that position: Now the spring being loosed, and suffered to act on a weight of 5 lbs.; it is required to determine the velocity generated in that weight, with the time of acceleration, for any space: the spring exerting its force horizontally?

SOLUTION, *by Mr.* GEO. HARVEY, *Jun. Plymouth.*

Let a ($= 6$ inches) denote the length which the spring contracts by the weight or force f ($= 64$lbs.), w the given weight upon which the spring acts ($= 5$lbs.), and $g = 16\frac{1}{12}$. Let x denote the distance described by the re-expansion of the spring, v the

v the velocity generated, and t the time of description: also let f' denote the motive force or weight which would compress the spring through the space $a - x$. Then by the nature of elasticity

$f : f' :: a : a - x$, whence $f' = (f \div a)(a - x)$; and therefore $(f' \div w) = (f \div aw)(a - x)$ expresses the accelerative force.

And by a known Theor. $v\dot{v} = \frac{2gf\dot{x}}{aw} \cdot (a - x)$, whence $v^2 = \frac{2gf}{aw}(2ax - x^2)$ which is correct, because when $x = 0$, $v = 0$; therefore

$v = \sqrt{\frac{2gf}{aw}(2ax - x^2)}$; and when $x = a$, $v = \sqrt{\frac{2gfa}{w}} = 172 \cdot 176$ inches.

Again $\dot{t} = \frac{\dot{x}}{v} = \frac{\dot{x}}{\sqrt{\{(2gf \div aw)(2ax - x^2)\}}} = \frac{\dot{x}}{\sqrt{(2ax - x^2)}}\sqrt{\frac{aw}{2gf}}$, and taking the fluents $t = R \times \sqrt{(aw \div 2gf)}$, where R denotes the length of a circular arc, radius 1, and versed-sine $x \div a$, which when $x = a$, becomes $\sqrt{(aw \div 2gf)} \times 1 \cdot 57079 = \cdot 054739$ seconds.

Solutions were also received from Messrs. Bazley, Marratt, Toplis, *and* Watts.

VII. QUESTION 247, *by* Amicus.

A vessel is to be made in the form of a segment of a sphere, capable of containing a given number of gallons. What must be its dimensions, so that being filled with water, the time of evacuation through a small aperture made in the centre of the bottom may be a minimum?

First Solution, *by Mr.* Thomas Bazley.

Put x for the height of the segment, and d for the diameter; also let $p = 3 \cdot 1416$, and $s =$ the solidity. Whence $p(dx - x^2)\dot{x} = \dot{s}$ and $s = p(\frac{1}{2}dx^2 - \frac{1}{3}x^3)$, and therefore $d = (6s + 2px^3) \div 3px^2$: now the area of the descending surface of the fluid is $p(dx - x^2) = p(\frac{6s + 2px^3}{3px} - x^2) = \frac{6s - px^3}{3x}$, and it is shewn by writers on fluxions that the time of evacuation is $\int \frac{-A\dot{x}}{2a\sqrt{(gx)}}$; where A is the area of the descending surface, and a that of the aperture which is supposed constant; wherefore, putting $(6s - px^3) \div 3x$ for A, we have $t = \int \dot{x}(px^3 - 6s) \div 6ax\sqrt{(gx)} =$ a minimum a minimum, and consequently $\dot{x}\,(px^3 - 6s) \div 6ax\sqrt{(gx)} = 0$, and therefore $px^3 = 6s$, or $s = \frac{1}{6}px^3$. But $\frac{1}{6}px^3$ is the solidity of a sphere whose diameter is x, and hence it appears that the entire sphere answers the conditions of the question.

SECOND SOLUTION, *by Mr.* WM. WATTS, *Plymouth*.

Put $r =$ the radius of the sphere, $x =$ the height of the descending surface above the orifice at the end of the time t, $A =$ the area of the surface at that time, $a =$ the area of the orifice, $g = 32\frac{1}{6}$ feet, $n = 3{\cdot}14159$, &c. and $s =$ the content of the segment. Then the velocity of the water through the orifice will be $= \sqrt{(2gx)}$, and the quantity discharged during the instant $\dot{t}$ will be $= a\dot{t}\sqrt{(2gx)}$. But in that instant the upper surface of the water descends through the space $\dot{x}$; therefore the quantity discharged in that time will be $= A\dot{x}$: Hence we have $a\dot{t}\sqrt{(2gx)} = -A\dot{x}$, or $\dot{t} = \dfrac{-A\dot{x}}{a\sqrt{(2gx)}}$.

When the solid is a sphere, as in the present example, A is $= n(2rx - x^2)$, and we have $\dot{t} = \dfrac{n}{a\sqrt{(2g)}} \times (2rx^{\frac{1}{2}}\dot{x} - x^{\frac{3}{2}}\dot{x})$, and (by integration) $t = \dfrac{n}{a\sqrt{(2g)}} \times (\frac{4}{3}rx^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}})$, the time of emptying the spherical segment whose height is x and radius r. Now, by the question, this value of t is to be a minimum; and the content of the segment, or the expression $\frac{1}{3}n(3rx^2 - x^3)$, a constant quantity (s); the radius r, and the height x of the segment, being both considered as variable quantities.

By taking the fluxion of $\frac{4}{3}rx^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}}$, the variable part of the expression for the time, we get $\dfrac{\dot{r}}{\dot{x}} = \dfrac{3x^{\frac{3}{2}} - 6rx^{\frac{1}{2}}}{4x^{\frac{3}{2}}} = \dfrac{3x - 6r}{4x}$.

And by taking the fluxion of $\frac{1}{3}n(3rx^2 - x^3) = s$, we get $\dfrac{\dot{r}}{\dot{x}} = \dfrac{x - 2r}{x}$, whence $\dfrac{3x - 6x}{4x} = \dfrac{x - 2r}{x}$, or $x = 2r$.

Whence it appears that a sphere of a given capacity will be emptied in a less time, than any spherical segment of the same capacity.

Solutions were also received from Messrs. Filmatt, Grisenthwaite, Harvey, Swale, *and* Toplis.

VIII. QUESTION 248, *by* PETER PUZZLE.

To cut a given square into seven pieces which may be put together one way, so as to form two unequal squares, and another way, so as to form two rectangles.

SOLUTION, *by Mr.* JOHN CAVILL.

Fig. 7. Pl. 1. From the opposite angles *a* and *c* of the square *abcd* draw *ax* and *cy*, bisecting the sides *bc* and *da* in *x* and *y*; from the other angles *b*, *d*, draw *bn* and *ds* parallel to each other and perpendicular to *cy* and *ax* respectively, cutting *ax* in *r* and *cy* in *m*; then if the square be cut through these lines it will be divided as required.

For it is plain from the construction, that if the parts BC′, B′C, and AA′ be joined by the sides *ad*, *bc*; *ab*, *cd*; *ar*, *cm*; they will form a square whose side will be *ar* or *cm* and $=$ 4 times the square D. Also AC, A′C′ and BB′ being joined by *ax*, *cy*; and *cx*, *ay*, will form two rectangles unequal and dissimilar.

Remark. If BC′, B′C, AA′ be joined by *ay*, *dy*; *bx*, *cx*; and *ax*, *cy* respectively, they will form three equal squares and a rectangle; and if *cx*, *ay*; *ax*, *cy* are joined, they will form two equal and similar rectangles.

IX. QUESTION 249, *by* AMICUS.

One end of a perfectly flexible chain is fastened at a given point, and the other end thrown over a pulley placed in a given position. It is required to find what part of the chain must be thrown over the pulley, so that the whole may rest in equilibrio, the length of the chain and its weight being given?

SOLUTION, *by* AMICUS, *the Proposer.*

In fig. 8. pl. 1. Let A be the given point where the end of the chain is fastened, BC the part which hangs over the small pulley, fixed at B, and AGB the catenary formed by the other part of the chain. Then the whole will evidently rest in equilibrio when the weight of the part BC is equal to the tension at B in the direction of the curve at that point. Let tangents BD and AD be drawn to the curve at B and A to intersect in D, and draw the vertical line DH. Then it is proved in several of the elementary books on mechanics, that the tension at B is to the weight of the chain BGA, as the sine of the angle ADH to the sine of the angle ADB; or, since the weights of any two portions of the chain are proportional to their lengths, the part which is equivalent to the tension at B is $= \frac{\text{length BGA} \times \text{sin. ADH}}{\text{sin ADB}}$.

Draw BE and AF parallel to the horizon, meeting the axis GE of the curve in E and F, and draw AK parallel to GE, meeting BE produced in K. Put BK $= n$. EF $= m$, BE $= y$, EG $= x$, arch BG $= z$, AF $= u$. GF $= v$, arch AG $= w$, the length of the chain $= l$, and the length of a portion of the chain whose weight is equal to the tension at G $= a$.

Then $\dot{y} \div \dot{z}$ is = the sine, and $\dot{x} \div \dot{z} =$ the cosine of the angle BDH, Also $\dot{u} \div \dot{w}$ is = the sine, and $\dot{v} \div \dot{w} =$ the cosine of the angle ADH; therefore the sine of the sum of these two angles, or of the angle

BDA, is $= \frac{\dot{y}\dot{x}}{\dot{z}^2} + \frac{\dot{w}\dot{v}}{\dot{w}^2}$, conseq. the part BC, (or $\frac{\text{length BGA} \, . \, \text{sin ADH}}{\text{sin ADB}}$)

$$\text{is} = (w + z)\frac{\dot{u}}{\dot{w}} \div \left(\frac{\dot{y}\dot{x}}{\dot{z}^2} + \frac{\dot{u}\dot{v}}{\dot{w}^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots \alpha$$

Now, by the property of the catenary, $2ax + x^2 = z^2$, and taking fluxions, we have $\dot{z} = \dot{x}(a + x) \div z$, or $\dot{x} \div \dot{z} = z \div (a + x)$. Also $\dot{y}^2 = \dot{z}^2 - \dot{x}^2 = \dot{x}^2 \left\{(a + x)^2 - z^2\right\} \div z^2 = \dot{x}^2 a^2 \div z^2$, or $\dot{y} = \dot{x}a \div z$, and $\dot{y} \div \dot{z} = a \div (a + x)$. In a similar way we find $\dot{u} \div \dot{w} = a \div (a + v)$, and $\dot{v} \div \dot{w} = w \div (a + v)$; these values being substituted in equation α, we have BC $= \frac{(w+z)(a+x)^2(a+v)}{z(a+v)^2 + w(a+x)^2}$,

and consequently $\frac{(w + z)(a + x)^2(a + v)}{z(a + v)^2 + w(a + x)^2} + w + z = l$.

Or, since $v = x - m$, we have

$$\frac{(w + z)(a + x)^2(a - m + x)}{z(a - m + x)^2 + w(a + x)^2} + w + z = l \ldots\ldots\ldots\ldots \beta$$

Again, by the property of the curve, $y = a \times \text{h.l.}\left\{(a + x + z) \div a\right\}$ and $u = a \times \text{h.l.}\left\{(a - m + x + w) \div a\right\}$; hence

$$y + u = n = a \times \text{h.l.}\frac{a + x + z}{a} + a \times \text{h.l.}\frac{a - m + x + w}{a} \ldots\ldots\ldots\ldots \gamma$$

Now, if in the equations β, γ we substitute for z and w their values $\sqrt{(2ax + x^2)}$ and $\sqrt{\left\{(2a - m + x)(x - m)\right\}}$, we shall have two equations containing only x, a, and known quantities, and from thence we can determine x and a (by the method of trial and error) when the given parts are expressed in numbers.

When

When m is $= 0$, or the points A and B are in a horizontal line, v is $= x$, $z = w$, and $y = \frac{1}{2}n$; and the expression for BC becomes $a + x$; hence we have $a + x + 2z = l$; also y, or $\frac{1}{2}n = a \times \text{h.l.} \left\{ (a + x + z) \div a \right\}$; and, passing to numbers,

$$e^{\frac{\frac{1}{2}n}{a}} = \frac{a + x + z}{a},$$ e being the number whose h. l. is 1.

Hence we have these two equations, $a + x + 2\sqrt{(2ax + x^2)} = l$, and $ae^{\frac{\frac{1}{2}n}{a}} = a + x + \sqrt{(2ax + x^2)}$, to determine the values of x and a.

If a portion of the chain, sufficient to preserve the equilibrium, be thrown over a pulley at A, instead of its end being fastened at that point; it is evident (when the points A and B are in a horizontal line) that the part hanging down from A will be equal to the part BC, or to $a + x$, and consequently we have

$$2(a + x) + 2z = l, \text{ and } ae^{\frac{\frac{1}{2}n}{a}} = a + x + z = \tfrac{1}{2}l.$$

Solutions were also received from Messrs. Harvey, Swale, Toplis, *and* Watts.

X. QUESTION 250, *by* JACK HAZARD.

Suppose the position of a ball to be given on a triangular billiard table; to find the direction in which it must be struck, so that after rebounding from all sides, it shall strike another ball whose position is likewise given: the angles of incidence and reflexion being equal.

Also, to determine the direction the ball must take, that when continually reflected from all the sides, it shall pursue the same track.

SOLUTION, *by Mr.* JOHN KAY, *Royton.*

Let ABC (fig. 9. pl. 1.) be the triangular table, P and Q the given points where the balls are placed. From P draw PG perpendicular to AB, making KG = KP; then G will be a given point. From G draw GH perpendicular to CB, making HL = GL; then H will be a given point. From Q draw QI perpendicular to AC, making IM = QM; then I will be a given point. Draw HI cutting BC in E, and AC in F; and GE cutting AB in D; join PD and QF; then if the ball at P, be struck in the direction PD, it will rebound from D to E, from E to F and from F to Q.

Because PK = KG, the right-angled triangles PKD, KGD are equal in every respect, therefore the angle PDK is equal to the angle KDG, or to the opposite angle EDB; consequently the ball after striking AB at D will be reflected in the direction DE.

Again because GL = HL, the angle GEL is equal to the angle HEL, or to the opposite angle FEC; therefore the ball after striking BC, at E, will proceed in the direction EF. Also because QM = MI, the angle QFM is equal to the angle IFM, or to the opposite angle CFE; therefore the ball after striking AC, at F, will proceed in the direction EQ.

Next to determine the direction the ball must take, from a given point, so that when continually reflected from all the sides it may proceed in the same track.

Draw *f*P*a* (fig. 10. pl. 1.) to make the angle P*a*A equal to the angle ACB which is opposite to the side AB. Also draw *ab* to make the angle B*ab* equal to the angle P*a*A, *bc* to make the angle C*bc* equal to the angle *ab*B, *cd* to make the angle A*cd* equal to the angle C*cb*, *de* to make the angle *ed*B equal to the angle *cd*A, and join *ef*. Then it is evident that, if the ball be struck at P in the direction P*a*, it will be reflected from *a* to *b*, from *b* to *c*, from *c* to *d*, and from *d* to *e*; and we have only to prove that it will be reflected from *e* to *f*, and from *f* to *a*.

Because the angles *fa*A, *ba*B are each equal to the angle ACB, it is obvious that the angles *ab*B, C*bc* are each equal to the angle BAC; the angles C*cb*, A*cd* each equal to the angle ABC; and the angles *cd*A, *ed*B each equal to the angle ACB: hence *cd* is parallel to *fa*, and *de* to *ab*; and the triangles *cd*A, *a*B*b* and *bc*C are equiangular,

therefore $cf : da :: c\mathrm{A} : \mathrm{A}d :: c\mathrm{C} : cb$,
$da : be :: a\mathrm{B} : \mathrm{B}b :: cb : b\mathrm{C}$,
and ex æquo $cf : be :: c\mathrm{C} : b\mathrm{C}$;
therefore *fe* is parallel to *cb*; consequently the angle C*ef* is equal to the angle B*ed*, and the angle C*fe* equal to the angle *af*A; or the ball after striking the side BC at *e*, will be reflected from *e* to *f*, and from *f* to *a*, and will continue to move in the track P*abcdef*P.

It may be shewn in the same way that, if the ball be struck in a direction parallel to *ab* or to *bc*, it will continue to move in the manner required. Hence if the ball be placed at a given point on the table, there are three directions in which it may be struck so that it shall continue to move in the same track, and return to the place from whence it set off after being twice reflected from each of the sides. It may, however, be placed in such a position as to return after one reflection; for if we suppose the lines

lines *fa*, *cd* and *ab*, *de* to coincide, or a triangle *abc* (fig. 11. pl. 1.) to be inscribed in the given one, so, that its sides may make equal angles with the sides AB, BC, CA, it is obvious that if the ball be placed in any of the sides *ab*, *bc*, *ca*, and struck in the direction of that side, it will continue to move in the track *abc*. The points *a*, *b*, *c* may be found by drawing the lines A*b*, C*a*, and B*c* perpendicular to the sides of the triangle; for then the points A, *c*, *b*, B are in a circle, and the points B, *a*, *c*, C are in a circle; hence the angles *bc*C, *ac*A are each equal to the angle at B, the angles *ab*B, *cb*C each equal to the angle at A, and the angles *ca*A, *ba*B each equal to the angle at C.

Solutions were also received from Messrs. Butterworth, Filmatt, Marratt, Reeves, Swale, *and* Toplis.

XI. QUESTION 251, *by* TRIGONOMETRICUS.

Let P be a given point in CF the diameter of a circle, and PAB any line meeting the circle in A and B: Then the sine of the difference of the arcs CA, CB, has a given ratio to the difference of their sines. Required the demonstration?

FIRST SOLUTION, *by Mr.* JOHN BUTTERWORTH, *Haggate.*

1st. When the point is without the circle. Fig. 12. pl. 1. Draw the sines AE and BH, and the radius BO. Make AI perpendicular to BH and AD to BO. Then BI is the difference of the sines, and AD the sine of the difference, of the arches CA, CB. Now the right-angled triangles ADK, BIK are evidently similar, and AK is parallel to PO; therefore BI : AD :: BK : AK :: BO : PO; but BO and PO are given lines, therefore BI has to AD a given ratio.

2nd. When the point is within the circle. Fig. 13. Pl. 1. Draw the diameter BOR, and produce the sine AE to meet the circle again in Q; join QR and QB, and draw QI perpendicular to BH and QD to BR. Then it is evident that the arch CQ is equal to the arch AC, therefore BI will be equal to the difference of the sines, and QD equal to the sine of the difference of the arches AC, CB. Now if the right angles BQR, IQA be taken from the angle BQA, there will remain the angle BQI = to the angle AQR = to the angle ABR; and since the angles at K and O are equal, the triangles QKB, OBP will be similar; and the triangles IBK and KDQ are also similar; therefore BI : QD :: BK : QK :: PO : OB; but PO and OB are given lines, therefore the ratio of BI to QD is given.

SECOND

SECOND SOLUTION, *by Mr.* J. H. SWALE, *Leeds.*

1st. When the point is within the circle. Fig. 14. Pl. 1.—Draw CD parallel to AB meeting the circle again in D; then the arch DB is = to the arch CA, and the arch CD = to the difference of the arches AC and CB. Draw DE and CK perpendicular to the radius BO, and DI perpendicular to CF; also draw DH parallel to BO meeting CO in Q and CK in H. Then CH is the difference of the sines, and DI the sine of the difference of the arches AC, CB. Now the right-angled triangles CHQ, QDI are evidently similar, and because CD is parallel to AB, and DQ to BO, the triangles CDQ, PBO are also similar; therefore $CH : DI :: CQ : QD :: PO : OB$, a given ratio.

2nd. When the point is without the circle. Fig. 15. Pl. 1. Draw AI perpendicular to the sine BH, and BD perpendicular to the radius AO, and join DI. Then BI is the difference of the sines, and BD the sine of the difference, of the arcs AC, CB; and because of the right-angles at D and I, the points A, D, I and B are in a circle, therefore the angle $IBD = IAD = AOP$; and the angle $IDB = IAB = OPA$; therefore the triangles IDB and OPA are similar, and $BI : BD :: AO : OP$, a given ratio.

THIRD SOLUTION, *by Mr.* BAZLEY.

Draw the sines AE, BH (fig. 15. pl. 1.) which suppose to lie on the same side as the point P in regard to the centre O. Let the distance OP be denoted by d, and, according to the usual notation of writers on trigonometry, let the greater arch CB be called A, and the lesser arch CA be called B. Hence $PE = d - \cos B$, and $PH = d - \cos A$, and therefore by the similar triangles PEA, PHB, $\sin A : d - \cos A :: \sin B : d - \cos B = \sin B\,(d - \cos A) \div \sin A$, and consequently $\cos B = d - \sin B\,(d - \cos A) \div \sin A$. Now by the well known formula in trigonometry $\sin(A - B) = (\sin A \cos B - \sin B \cos A) \div R$, in which if we substitute for cos B its value as above, we get

$$\sin(A-B) = \left\{\sin A \times d - \sin B\,(d - \cos A) - \sin B \cos A\right\} \div R$$
$$= (d \sin A - d \sin B) \div R = (d \div R)(\sin A - \sin B).$$

Therefore $\sin(A - B) : \sin A - \sin B :: d : R$, a constant ratio.

In whatever part of the diameter the point P falls the demonstration will be the same, attending to the changes of the signs. Thus in OF the cosines must be esteemed negative, and also the sines are negative below CF.

Solutions were also received from Messrs. Cavill, Filmatt, Kay, Reeves, Smith, Toplis, *and* Trigonometricus Cantabrigiensis.

XII.

XII. QUESTION 252, *by* Amicus.

To find a curve, such, that the rectangle contained by the radius of curvature and the abscissa shall always be equal to a given space?

Solution, *by the Rev.* John Toplis, *Nottingham.*

Let x and y denote the co-ordinates of the curve, and a^2 the given space. Then $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y}$ will be the expression for the radius of curvature, and by the question we have

$$\frac{(dx^2+dy^2)^{\frac{3}{2}}x}{dxd^2y}=a^2.$$

Assume $pdx=dy$, then, as dx is supposed constant, we have $dpdx=d^2y$; substitute these values of dy and d^2y in the above equation and it becomes

$\frac{xdx}{a^2}=\frac{dp}{(1+p^2)^{\frac{3}{2}}}$ whose integral is $\frac{x^2}{2a^2}=\frac{p}{\sqrt{(1+p^2)}}$;

hence $p=\frac{x^2}{\sqrt{(4a^4-x^4)}}$, and $y=\int pdx=\int\frac{x^2dx}{\sqrt{(4a^4-x^4)}}$, which expression is the equation to the common elastic curve, and may be integrated by means of elliptic arches.

The problem may be resolved in the same way when the radius of curvature is any given function of the ordinate x.

Mr. Filmatt *answered this question.*

XIII. QUESTION 253, *by Mr.* Cunliffe.

Let two inflexible rods of given lengths, considered without weight, be connected by a joint; and let given weights be fastened, one at the joint, and the other two at the extremities of the rods; to determine how the rods must be placed at two given points upon an horizontal rod or line, so as to rest in equilibrio?

First Solution, *by Mr.* Cunliffe, *the Proposer.*

The following solution will shew that this is an indeterminate problem.

Any point, within proper limits, may be assumed at pleasure upon one of the rods by means of which another point may be determined upon the other rod, which point, so determined, and the assumed point, being placed at the given points upon the horizontal rod, the weights will rest in equilibrio when the rods connected with the weights are in an horizontal position.

Let AB, AC (fig. 16. pl. 1.) represent the two rods, whose lengths are given, connected by a joint at A: Let A denote the weight

weight fastened at the point A; B and C the other two weights fastened at the other ends of the rods. Let P represent the point assumed upon AB, and P′ the point to be determined upon AC, also let G represent the centre of gravity of the weights A and B; and G′ that of the weights A and C. Put $AB = a$, $AC = b$, $AP = z$, and $AP' = v$. Then by the known property of the lever, $B \times PB \div AP = B \times (a - z) \div z$ will express that part of the weight A which is balanced by B; and therefore $A - B(a - z) \div z = \{(A + B)z - Ba\} \div z$ will express the part of the weight A which remains to be balanced by C. Again by the property of the lever we shall have $\{(A + B)z - Ba\} \times v \div z = C(b - v)$ which gives $v = Cbz \div \{(A + B + C)z - Ba\}$.

Remark. It is very obvious that AP cannot be assumed less than AG.

SECOND SOLUTION, *by the Rev.* J. TOPLIS.

Let B (fig. 17. pl. 1.) be the weight at the joint, A and C those at the ends of the rods. Take $BE : BC :: C : B + C$, and $BD : BA :: A : A + B$; then if AE, CD are respectively joined, the centre of gravity of the three weights will be at the intersection of the lines AE, CD or Q, and it is evident that if the horizontal line passes through Q the weights will be in equilibrio. Let L and G be the given points upon which the rods are to rest, and draw BI, AK, and CH perpendicular to LG; then by the property of the centre of gravity, $B \times BI = A \times AK + C \times CH$. Put $AB = a$, $BC = b$, $BL = x$, and $BG = y$; then by similar triangles
$BL : BI :: AL = (a - x) : AK = BI \times (a - x) \div x$, and
$BG : BI :: CG = (b - y) : CH = BI \times (b - y) \div y$; hence
$B \times BI = A \times BI \times \frac{a - x}{x} + C \times BI \times \frac{b - y}{y}$, or $B = \frac{Aa}{x} - A + \frac{Cb}{y} - C$; therefore by reduction, $y = \frac{Cbx}{(A + B + Cx) - Aa}$.

Whence it is evident that if L be taken any where between D and A, then BG is $= \frac{Cb \times BL}{(A + B + C)BL - Aa}$; so that the rods may be placed in an indefinite number of positions.

Solutions were also received from Messrs. Butterworth, Filmatt, Kay, M'Doneld, *and* Swale.

XIV.

XIV. QUESTION 254, *by* QUIDAM.

Find the sum of the infinite series

$$\frac{3}{4.5.3^2} + \frac{5}{5.6.3^4} + \frac{7}{6.7.3^6} + \frac{9}{7.8.3^8} + \&c.$$

FIRST SOLUTION, *by Mr.* JONATHAN REEVES.

By division $\frac{1}{(1-x)^2}$ is $= 1 + 2x + 3x^2 + 4x^3 + \&c.$

Hence $\frac{3x^3 - x^4}{(1-x)^2}$ is $= 3x^3 + 5x^4 + 7x^5 + 9x^7 + \&c.$

Multiplying by $\dot{x}$ and taking the fluents, we have

$$\frac{2}{1-x} - 2 + 3x + \frac{x^2}{2} - \frac{x^3}{3} - 5\text{h. l. } \frac{1}{1-x} = \frac{3x^4}{4} + \frac{5x^5}{5} + \frac{6x^6}{6} + \&c.$$

Multiply by $\dot{x}$, take the fluents, and divide by x^4; then

$$\frac{7-5x}{x^4}\text{h.l.}\frac{1}{1-x} - \frac{7}{x^3} + \frac{3}{2.x^2} + \frac{1}{2.3x} - \frac{1}{3.4} = \frac{3x}{4.5} + \frac{5x^2}{5.6} + \frac{7x^3}{6.7} + \&c.$$

which agrees with the proposed series when $x = \frac{1}{3^2}$, and in this case the value of the expression is $= 42282$ h. l. $(9 \div 8) - 5103 + 121{\cdot}5 + 1{\cdot}5 - \frac{1}{12} = {\cdot}01898028$ the sum required.

SECOND SOLUTION, *by Mr.* ROBERT FILMATT.

The general term of the series is evidently $\frac{2x+1}{(x+3)(x+4)3^{2x}}$, which is equal to $\frac{2}{(x+3)^{2x}} - \frac{7}{(x+3)(x+4)3^{2x}}$. Therefore the given series is equal to the difference of the two series

$$2\left(\frac{z}{4} + \frac{z^2}{5} + \frac{z^3}{6} + \frac{z^4}{7} + \&c.\right) \text{ and}$$

$$7\left(\frac{z}{4.5} + \frac{z^2}{5.6} + \frac{z^3}{6.7} + \frac{z^4}{7.8} + \&c.\right) \text{ where } z = \frac{1}{3^2}.$$

Put $s = \frac{z}{4} + \frac{z^2}{5} + \frac{z^3}{6} + \frac{z^4}{7} + \&c.$

Then $sz^3 = \frac{z^4}{4} + \frac{z^5}{5} + \frac{x^6}{6} + \frac{z^7}{7} + \&c.$

And adding $z + \frac{z^2}{2} + \frac{z^3}{3}$ to both sides, we have $s\dot{z}^3 + z + \frac{z^2}{2} + \frac{z^3}{3} = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \frac{z^5}{5} + \&c. = \int \frac{\dot{z}}{1-z} =$ h.l. $\frac{1}{1-z}$. Therefore $s = \frac{1}{z^3}$.h.l. $\frac{1}{1-z} - \frac{1}{z^2} - \frac{1}{2z} - \frac{1}{3} =$ (when $z = \frac{1}{3^2}$) 729 hyp. log. $\frac{9}{8} - 81 - \frac{9}{2} - \frac{1}{3} = \cdot 0304996\bar{6}$.

Again $\frac{s}{z} - \frac{1}{4} = \frac{z}{5} + \frac{z^2}{6} + \frac{z^3}{7} + \&c.$ and by subtraction $s - \frac{s}{z} + \frac{1}{4} = \frac{z}{4 \cdot 5} + \frac{z^2}{5 \cdot 6} + \frac{z^3}{6 \cdot 7} + \frac{z^4}{7 \cdot 8} + \&c.$ Therefore the sum required is $= 2s - 7(s - \frac{s}{z} + \frac{1}{4}) = 58s - \frac{7}{4} = \cdot 01898028$.

THIRD SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Put hyp. log. $\frac{1}{1-x} = s = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c. \ldots\ldots (1)$

then $sx = x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{4} + \&c. \ldots\ldots (2)$

and taking the difference of (1) and (2) and multiplying by 6

$$6s(x-1) = -6x + \frac{6x^2}{1.2} + \frac{6x^3}{2 \cdot 3} + \frac{6x^4}{3 \cdot 4} + \frac{6x^5}{4 \cdot 5} + \&c. \ldots\ldots\ldots\ldots (3)$$

also by adding (1) and (2) we have

$$s(x+1) = x + \frac{3x^2}{1.2} + \frac{5x^3}{2 \cdot 3} + \frac{7x^4}{3 \cdot 4} + \frac{9x^5}{4 \cdot 5} + \&c. \ldots\ldots\ldots\ldots (4)$$

and taking the difference of (3) and (4)

$$s(7-5x) = 7x - \frac{3x^2}{1.2} - \frac{x^3}{2.3} + \frac{x^4}{3 \cdot 4} + \frac{3x^5}{4 \cdot 5} + \frac{5x^6}{5 \cdot 6} + \frac{7x^7}{6.7} + \&c.$$

and transposing and dividing by x^4 we have

$$\left\{ s(7-5x) - 7x + \frac{3x^2}{1.2} + \frac{x^3}{2 \cdot 3} - \frac{x^4}{3 \cdot 4} \right\} \div x^4 = \frac{3x}{4 \cdot 5} + \frac{5x^2}{5 \cdot 6} + \frac{7x^3}{6.7} + \&c.$$

By taking $x = \frac{1}{3^2}$ this last formula will coincide with the case of the question, and we shall have

$$\tfrac{1}{12}(507384s - 59761) = \tfrac{1}{12}(507384.\text{h.l.}(9 \div 8) - 59761) = \cdot 01898028$$
$$= \frac{3}{4 \cdot 5 \cdot 3^2} + \frac{5}{5 \cdot 6 \cdot 3^4} + \frac{7}{6 \cdot 7 \cdot 3^6} + \frac{9}{7 \cdot 8 \cdot 3^8} + \&c.$$

XV.

XV. QUESTION 255, *by Mr.* CUNLIFFE.

To determine the greatest space that can possibly be included within two equal given right lines, and a curve, whose length is equal to one of the given lines?

SOLUTION, *by* AMICUS,

Fig. 18. Pl. 1. Let O be a given point, and OA and OB given lines, to find the nature of the curve APB so that the area OAPB may be a maximum when the arch APB is of a given length.

Draw any line OP, and put OP $= y$, and the arch $ap = x$, the radius Oa being 1; then the fluxion of the area OAP is $= \frac{1}{2}y^2\dot{x}$, and the fluxion of the arch AP $= \sqrt{(\dot{y}^2 + y^2\dot{x}^2)}$. Taking the fluxions of these expressions, supposing $\dot{x}$ only to be variable, and multiplying by the constant but indeterminate quantities e and f, according to the rule on page 101 of Simpson's Miscellaneous Tracts, we have (after dividing by $y^2\ddot{x}$), $\frac{e}{2} + \frac{f\dot{x}}{\sqrt{(\dot{y}^2 + y^2\dot{x}^2)}} = 0$, or putting $d = \frac{2f}{e}$, and reducing, we get $\dot{x} = \frac{\dot{y}}{\sqrt{(d^2 - \ {}^2)}}$, the fluent of which is $x =$ arch sin $\frac{y}{d} + c$.

Now when $x = 0$, $y =$ OA $= a$, therefore $c = -$ arch sin $\frac{a}{d}$, and $x +$ arch sin $\frac{a}{d} =$ arch sin $\frac{y}{d}$.

This shews that the curve is a circle passing through the points O, A, B and having d for its diameter. For if OE be the diameter, and OS be perpendicular to OE, and the sines aH, pQ be drawn perpendicular to OS, then by similar triangles, OE(d) : OP(y) :: Op(1) : pQ $= y \div d$, and OE(d) : OA(a) :: Oa(1) : aH $= a \div d$; hence, arch ap + arch aS (whose sine is aH) $=$ arch pS (whose sine is pQ), or $x +$ arch sin $\frac{a}{d} =$ arch sin $\frac{y}{d}$.

The value of d may be found by the method of trial and error when the lines OA, OB and the arch AB are given in numbers; For by assuming two values of d, we may find the corresponding values of the arches AE and BE, and by comparing their sum with the length of the arch AB, and noting the errors, a nearer value of d may be obtained. And by making fresh assumptions and repeating the process we may find the value of d to any degree of accuracy that we please.

The diameter being known, the area is easily found, for the space OAB is equal to the sector ADB, and the two triangles AOD and ODB.

But when the lines OA, OB and the arch AB are equal, as in the question, the diameter may be found by a simpler process. For AB will then be bisected in E, and the arch AE will be = ½AO, = to the sine of half the supplement of AE. To find an arch z whose length shall be equal to the sine of half its supplement, the radius being 1, let n be the number of degrees in the arch z, and $p = 3{\cdot}1415926$ the length of a semi-circle, or 180°.

Then $180^\circ : n :: p : z = \frac{pn}{180} = \frac{n}{57{\cdot}295779}$, or $\log. z = \log. n - 1{\cdot}7581226$. It is evident that the arch z is less than the radius, or n less than 57°: suppose $n = 52^\circ$, then half the supplement is $= 64^\circ$, and we have

l. 52° = ...	1·7160033
subtract	1·7581226
l. arch z = ...	9·9578807
l. sin 64°	9·9536602
the error = ...	+ 42205

Hence it appears that n is rather less than 52°.

Suppose then that $n = 51^\circ$, then half the supplement is $= 64^\circ\ 30'$, and we have

l. 51° = ...	1·7075702
subtract	1·7581226
l. arch z = ...	9·9494476
l. sin 64° 30′ = ...	9·9554882
the error = ...	— 60406
	42205
The sum of the errors = ...	102611

So that n is greater than 51° but less than 52°.

Hence $102611 : 42205 :: 1^\circ : 24'..40''$, and n (or the angle ADE) $= 52^\circ - (24'..40'') = 51^\circ..35'..20''$ very near.

Put $\varphi =$ the angle AOD $= \frac{1}{2}$ADE, and $r =$ AD $= \frac{1}{2}d$; then $r = \frac{a}{2 \cos \varphi}$, and the area of the two triangles ADO, BDO $= r^2 \sin 2\varphi = \frac{a^2 \sin \varphi}{2 \cos \varphi}$; also the area of the sector ABD is

$s = r \times AE = r \times \frac{a}{2} = \frac{a^2}{4\cos\varphi}$: Hence the whole area $OAB = \frac{a^2(1+2\sin\varphi)}{4\cos\varphi} = a^2 \times \cdot 519312$.

Solutions were also received from Messrs. Crane, Grisenthwaite, Harvey, Marratt, Swale, Toplis, *and* Watts.

XVI. QUESTION 256, *by* DIOPHANTUS.

To find x, y, and z, in whole numbers when $xy + z^2$, $xz + y^2$, $yz + x^2$, and $(xy + z^2)^{\frac{1}{2}} + (xz + y^2)^{\frac{1}{2}} + (yz + x^2)$ are four squares.

FIRST SOLUTION, *by Mr.* CUNLIFFE.

Put $x = 4(y + z)$; then $xy + z^2 = 4y(y + z) + z^2 = (2y + z)^2$; whence $(xy + z^2)^{\frac{1}{2}} = 2y + z$. Also $xz + y^2 = 4z(y + z) + y^2 = (y + 2z)^2$; whence $(xz + y^2)^{\frac{1}{2}} = y + 2z$. And $yz + x^2 = yz + 16(y + z)^2 = 16y^2 + 33yz + 16z^2 =$ a square $= (rz - 4y)^2 = r^2z^2 - 8rzy + 16y^2$; whence $y = \frac{z(r^2 - 16)}{8r + 33}$; $x = 4(y + z) = \frac{4z(r^2 + 8r + 17)}{8r + 33}$.

Again we shall have $(xy + z^2)^{\frac{1}{2}} + (xz + y^2)^{\frac{1}{2}} + (yz + x^2)^{\frac{1}{2}} = 3y + 3z + rz - 4y = z(r + 3) - y = z(r + 3) - \frac{z(r^2 - 16)}{8r + 33} = \frac{z(7r^2 + 57r + 115)}{8r + 33} =$ a square $= n^2$; whence $z = \frac{n^2(8r + 33)}{7r^2 + 57r + 115}$.

Put $n = s(7r^2 + 57r + 115)$, then $z = s^2(7r^2 + 57r + 115)(8r + 33)$, $y = z(r^2 - 16) \div (8r + 33) = s^2(7r^2 + 57r + 115)(r^2 - 16)$, $x = 4z(r^2 + 8r + 17) \div (8r + 33) = 4s^2(7r^2 + 57r + 115)(r^2 + 8r + 17)$; where r and s may be taken at pleasure.

Take $s = 1$ and $r = 5$, then $z = 575 \times 73 = 41975$, $y = 575 \times 9 = 5175$ and $x = 4(y + z) = 188600$, which are three numbers that will answer. But each of the three preceding numbers is divisible by 25, therefore we may take $z = 1679$, $y = 207$ and $x = 7544$.

SECOND SOLUTION, *by Mr.* W. WALLACE, *R. M. College.*

Assume $xy + z^2 = (p + z)^2$, and $xz + y^2 = (q + y)^2$.
Then $xy = p(p + 2z)$, $xz = q(q + 2y)$;

and

and hence, eliminating x

$$pz(p+2z)=qy(q+2y).$$

This equation will be immediately satisfied if we make

$$pz=qy, \text{ and } p+2z=q+2y,$$

and from this assumption we get $p=2y$, $q=2z$; and since $xy=p(p+2z)=2y(2y+2z)$, therefore $x=4(y+z)$. We have now to make $yz+x^2$ or its equal $yz+16(y+z)^2$ a square: Let us make

$$yz+16(y+z)^2=r^2,$$

then, transposing and resolving into factors, we get

$$yz=(r+4y+4z)(r-4y-4z).$$

Assume now $my=n(r+4y+4z)$,

Then $nz=m(r-4y-4z)$.

From these equations we get

$$y=\frac{(8mn+n^2)r}{4m^2+mn-4n^2},\ z=\frac{(m^2-8mn)r}{4m^2+mn-4n^2},$$

and substituting in the equation $x=4(y+z)$, we have $x=\frac{4(m^2+n^2)r}{4m^2+mn-4n^2}$. These values of x, y and z satisfy the three first conditions of the question; but from the nature of the functions to be made squares they will be equally satisfied although we reject the common denominator, therefore we have more simply

$$x=4(m^2+n^2)r,\ y=(8mn+n^2)r,\ z=(m^2-8mn)r$$

From these we find

$$xy+z^2=r^2(m^2+8mn+2n^2)^2.$$
$$xz+y^2=r^2(2m^2-8mn+n^2)^2,$$
$$yz+x^2=r^2(4m^2+mn-4n^2)^2;$$

And the sum of the square roots of these expressions is

$$r(7m^2+mn-n^2),$$

which must be made a square. Now this may be done in two ways.

FIRST. We shall evidently make it a square if we assume $r=7$, and give to m and n such values that $7m^2+mn-n^2=7v^2$. From this equation, by resolving into factors we have

$$7(m+v)(m-v)=n(n-m).$$

Now, putting a and b for any two numbers, we may assume

$$7b(m+v)=a(n-m),\ a(m-v)=bn,$$

and hence we get

$$m=\frac{(7b^2+a^2)v}{a^2-7b^2-ab},\ n=\frac{(14ab+a^2)v}{a^2-7b^2-ab}.$$

These values of m and n will be integral if we suppose v equal to the common denominator. From this method of resolution then we get the following formulæ for determining the numbers sought.

Take

Take $m = a^2 + 7b^2$, and $n = a^2 + 14ab$, a and b being any two numbers whatever, or m and n may be any two numbers in the proportion of $a^2 + 7b^2$ to $a^2 + 14ab$, then $x = 28(m^2 + n^2)$, $y = 7(8mn + n^2)$, $z = 7(m^2 - 8mn)$. These values of x, y and z will give us the values of $xy + z^2$, $xz + y^2$, $yz + x^2$ as already stated, (r being put $= 7$) and $\sqrt{(xy + z^2)} + \sqrt{(xz + y^2)} + \sqrt{(yz + x^2)}$ equal to $7^2(a^2 - ab - 7b^2)^2$.

SECONDLY. We may also render the function $r(7m^2 + mn - n^2)$ a square as follows. Let any numbers whatever be substituted for m and n in the expression $7m^2 + mn - n^2$, and let the result be put under this form, αv^2; then if we take $r = \alpha$, $r(7m^2 + mn - n^2)$ will become $r^2 v^2$, and consequently will be a square; we shall then have

$x = 4\alpha(m^2 + n^2)$, $y = \alpha(8mn + n^2)$, $z = \alpha(m^2 - 8mn)$.

But in order to make z positive we must make $m > 8n$.

Ex. 1. Let us suppose $m = 9$, $n = 1$.

Then $7m^2 + mn - n^2 = 23 \times 5^2$; therefore $\alpha = 23$, and hence $x = 4 \times 23 \times 82 = 7544$, $y = 23 \times 73 = 1679$, $z = 23 \times 9 = 207$.

Probably these are the smallest positive numbers that answer the conditions of the question.

Ex. 2. Let $m = 10$, $n = 1$. Then $7m^2 + mn - n^2 = 709$, a prime number, which can only be expressed under the prescribed form thus 709×1^2, here therefore $\alpha = 709$, and $x = 286436$, $y = 57429$, $z = 14180$.

XVII. QUESTION 257, *by Mr.* W. WALLACE.

Let APBC be a parallelogram. Draw AB, one of the diagonals, and from P, draw any line cutting the diagonal in E, and the other two sides of the parallelogram in D and *d*. Then ED × E*d* = PE². Required the demonstration?

FIRST SOLUTION, *by Mr.* ROBERT PARKINSON BROOKE.

Fig. 19. Pl. 1. The triangles BE*d*, AEP; and PEB, AED are similar, therefore PE : E*d* :: AE : EB; and ED : PE :: AE : EB; therefore, by equality, PE : E*d* :: ED : PE; or PE² = E*d* × ED.

SECOND SOLUTION, *by Mr.* J. T. M'DONELD.

Draw CF (fig. 19. pl. 1.) parallel to EP and equal to it. Then by similar triangles DE : CF = AE : AF = BF : BE = CF : *d*E.

Hence ED × E*d* = CF² = PE².

Solutions were also received from Messrs. Bazley, Butterworth, Buxton, Cavill, Filmatt, Kay, Marratt, Reeves, Smith, Swale, Toplis, Tyrotatos, *and* J. Wallace.

XVIII. QUESTION, 258, *by* AMICUS.

Find the position of a plane passing through a given point on the surface of a given right cone, so that the periphery of the ellipse made by the intersection of the plane with the conical superficies may be a minimum?

SOLUTION, *by* AMICUS, *the Proposer.*

Let aCb (fig. 20. pl. 1.) be the given cone, and B the given point on its surface. Let a plane be drawn through B, and the axis of the cone, to meet the conic surface in the straight lines aC, bC, and draw BA parallel to the base of the cone. Draw BD perpendicular to Ca, and let $BcdD$ be the section whose transverse axis is BD. Then since BD is the shortest line that can be drawn from B to Ca, it is obvious that if any plane be drawn to meet Ca, in a point below D, the transverse and conjugate axes of the section will be respectively greater than those of the section $BcdD$; therefore the periphery of the former will be greater than the periphery of the latter; and consequently the cutting plane cannot meet Ca below the point D when the periphery of the section is a minimum.

Let E, then, be the point where the cutting plane meets Ca; then it is pretty obvious that BE must be the transverse axis of the ellipse. Bisect BE in O, and let IH be the conjugate to BE; also draw EF and LOM parallel to AB. Put $AB = a$, $AC = CB = b$, $DB = c$, $DC = d$, $BE = x$, $IH = y$, $1 - (y^2 \div x^2) = e$, and $n = 3{\cdot}14159$, &c.

Then $DE = \sqrt{(x^2 - c^2)}$, $CE = d - \sqrt{(x^2 - c^2)}$, and by similar triangles AC : AB :: CE : EF, or

$b : a :: d - \sqrt{(x^2 - c^2)} : (a \div b)(d - \sqrt{(x^2 - c^2)}) = EF$. But $IH^2 = 4IO^2 = 2LO \cdot 2OM = AB \cdot EF = (a^2 \div b)(d - \sqrt{(x^2 - c^2)}) = y^2$:

$$\text{therefore } 1 - \frac{y^2}{x^2} = 1 - \frac{a^2}{bx^2}\left(d - \sqrt{(x^2 - c^2)}\right) = e,$$

and taking fluxions, we have

$$\frac{\dot{x}}{x}\left\{\frac{2a^2}{bx^2}\sqrt{(d - \sqrt{(x^2 - c^2)})} + \frac{a^2}{b\sqrt{(x^2 - c^2)}}\right\} = \dot{e}.$$

Now by Art. 434, Simpson's Fluxions, the periphery of the ellipse whose transverse axis is x and conjugate axis y is =

$$nx\left(1 - \frac{e}{2^2} - \frac{3e^2}{2^2.4^2} - \frac{3^2.5\,e^3}{2^2.4^2.6^2} - \&c.\right)$$

$$= nx\left(1 - Ae - Be^2 - Ce^3 - \&c.\right), \text{ putting } A = \frac{1}{2^2},$$

$B = \frac{3}{2^2.4^2}$, $C = \frac{3^2.5}{2^2.4^2.6^2}$, &c.

And

And this expression is to be a minimum; hence by taking fluxions and dividing by n, we have

$$\dot{x}\,(1-Ae-Be^2-Ce^3-\&c.)+x\,\dot{e}(-A-2Be-3Ce^2-\&c.)=0.$$

And if we substitute for $\dot{e}$, its value found above, and divide by $\dot{x}$ we shall have

$$(1-Ae-Be^2-Ce^3-\&c.)$$
$$+\left\{\frac{2a^2}{bx^2}\sqrt{(d-\sqrt{(x^2-c^2)})}+\frac{a^2}{b\sqrt{(x^2-c^2)}}\right\}\cdot\left\{-A-2Be-3Ce^2-\&c.\right\}=0.$$

From this equation and the preceding one which expresses the relation between x and e, we may easily determine the value of x by the method of trial and error.

If the values of c and d are such as to give $x=b$, the problem is impossible. Or if, in this case, φ be put to denote the angle aCb, then when the angle of the cone is greater than φ, the problem cannot possibly admit of a minimum, but the perimeter of the ellipse will continue to increase from the vertex downwards.

Messrs. Grisenthwaite *and* Swale *answered this question.*

XIX. QUESTION 259, *by Mr.* LOWRY.

If from any two points in the periphery of an ellipse, straight lines are drawn parallel to the axes of the curve, meeting the circles described on these axes in the points a, b; A, B; the lines ab, AB, have the same proportion as the axes of the ellipse.—Required the demonstration?

SOLUTION, *by Mr.* JOHN BUTTERWORTH.

From any two points P and S (fig. 21. pl. 1.) in the ellipse, let the lines PC, SD and PF, SE be drawn parallel to the semi-axes GO and IO, meeting them in C, D and E, F. Join Aa, Bb, Oa and Ob. Then by the property of the ellipse (see Lemma to Question 210), SE (DO) : AS :: IO : OG — IO :: aD : Sa; therefore the right-angled triangles ASa, ODa are similar, consequently the angle AaS = the angle OaD, and the points A, a, O are in a straight line. Again PF (OC) : BP :: IO : OG — IO :: Cb : Pb; therefore the right-angled triangles ObC and PBb are similar, and the points B, b, O are in a straight line. Hence the isosceles triangles Oab, OAB, are similar, and we have AB : ab :: AO : aO :: the axis major : the axis minor.

Solutions were also received from Messrs. Harvey, Kay, M'Doneld, Swale, *and* Toplis.

XX. QUESTION 260, *by* MECHANICUS.

If the ends of a perfectly flexible and inextensible string of a given length be fastened to two given points, and a small heavy body or bead descend freely along the string by the force of gravity; it is required to determine the circumstances of the motion and the time of descending through a given portion of the string?

SOLUTION, *by* MECHANICUS, *the Proposer*.

Let P and Q (fig. 22. pl. 1.) be the given points to which the ends of the thread are fastened; then the body will evidently describe a portion of an ellipse of which the given points are the foci. Let this ellipse be described and draw a line perpendicular to the horizon to touch the curve at L; then if the body descend from L, or from any point in the curve below L, it will evidently be retained in the curve by the re-action of the string; but if it descend from any point above L, it will leave the ellipse, and descend in a vertical direction till it meet the curve again in some point below L. Let, therefore, the point C be the place of the body at the commencement of the motion, and E its place at the end of any indefinite time t. Draw SE to bisect the angle PEQ and let Eg, drawn in a vertical direction, represent the force of gravity; decompose this force into two others, gk and Ek, one perpendicular to SE, and the other in the direction SE. Then it is plain that the force Ek is destroyed by the resistance of the string, and that the force gk only, is employed in accelerating the motion of the body. Therefore, because the tangent to the curve at E is perpendicular to SE, the body is accelerated in the direction of the tangent by the force gk; or it is accelerated in the same manner as if it descended freely along the curve CED without having any connection whatever with the string.

Let O be the centre of the ellipse and draw the vertical line OP′. Also draw OZ, CD, and EF parallel to the horizon; and let OH, the semi-conjugate to OZ, meet CD in I, and EF in K; then since the ellipse is given in magnitude and position, the semi-diameters OZ and OH are given. Also the angle OKF and the lines OD and OP′ are given.

Put OZ $= m$, OH $= n$, OD $= h$, OP′ $= a$, sin OKF $= s$, cos OKF $= c$, OF $= y$, EF $= x$, P′F $= u$, arch CE $= z$, the velocity acquired in descending through the arch CE, or through the height ME $= v$, and the force of gravity $= g$. Then we have $v = \sqrt{2g} \times \text{ME} = \sqrt{2g}\,(y - h)$. And by trigonometry OK $= y \div s$, and KF $= cy \div s$; also by the property of the

the ellipse $OH^2 : OZ^2 :: OH^2 - OK^2 : EK^2$, or

$$n^2 : m^2 :: n^2 - \frac{y^2}{s^2} : EK^2 = \frac{m^2}{n^2}\left(n^2 - \frac{y^2}{s^2}\right);$$

hence $EF = x = \frac{c}{s}y + \frac{m}{n}\sqrt{\left(n^2 - \frac{y^2}{s^2}\right)}$, and $P_{,}F = u = a - y$. And by taking fluxions, and reducing, we shall have

$$\dot{z} = \sqrt{(\dot{x}^2 + \dot{u}^2)} = \dot{y}\sqrt{\left\{\left(\frac{c}{s} - \frac{my}{ns\sqrt{(n^2s^2 - y^2)}}\right)^2 + 1\right\}},$$

and $t = \int\frac{\dot{z}}{v} = \int\frac{\dot{y}}{\sqrt{2g\,y-h)}} \times \sqrt{\left\{\left(\frac{c}{s} - \frac{my}{ns\sqrt{(n^2s^2 - y^2)}}\right)^2 + 1\right\}}$, which expression being integrated by infinite series and made to vanish when $y = h$, we shall obtain the requisite time for passing over any given space.

When the given points are in a horizontal line, c is $= 0$, $s = 1$, $m =$ the semi-transverse, $n =$ the semi-conjugate, and we have $t = \int\frac{\dot{y}}{\sqrt{2g(y-h)}} \times \frac{1}{n}\sqrt{\frac{n^4 + (m^2 - n^2)y^2}{n^2 - y^2}}$; and when $m = n$; or the curve is a circle, we have $t = \int\frac{n\dot{y}}{\sqrt{2g(y-h)}\sqrt{(n^2 - y^2)}}$ as it ought to be.

If r be put for the radius of curvature at E, the pressure against the curve, or the force exerted against the string in the direction SE, is $= g \times \left\{\frac{\dot{x}}{\dot{z}} + \frac{2(y-h)}{r}\right\}$. Let P denote this force, and substitute for $\dot{x}$ and $\dot{z}$ their values found above, then we shall have

$$P = g \times \frac{\frac{c}{s} - \frac{my}{ns\sqrt{(n^2s^2 - y^2)}}}{\sqrt{\left\{\left(\frac{c}{s} - \frac{my}{ns\sqrt{n^2s^2 - y^2)}}\right)^2 + 1\right\}}} + \frac{2g(y-h)}{r}.$$

And when the points are in a horizontal line

$P = \frac{gmy}{\sqrt{\{n^4 + (m^2 - n^2)y^2\}}} + \frac{2g(y-h)}{r}$. In this case if $y = n = a$, and $h = 0$, or the body descend through a quadrant of the ellipse, P will be $= g\left(1 + \frac{2n}{r}\right)$: and when m is $= n$, P will be $= 3g$, which we know to be true from other principles.

The Rev. Mr. Toplis *answered this question.*

XXI. QUESTION 261, *by* Amicus.

Let there be an indefinite number of circles PAB, PA′B′, PA″B″, &c. touching each other at P; it is required to find the nature of the curve cutting these circles in the points B, B′, B″, &c. so, that joining PB, PB′, PB″, &c. the area of the circular segments PAB, PA′B′, PA″B″, &c. may be equal to the same given space?

First Solution, *by Mr.* Lowry, *R. M. College.*

Let the line PQ (fig. 23. pl. 1.) be drawn through the centres of all the circles, and let B B′ B″ be the curve required, such, that if any circle be described through P to intersect the curve in B and have its centre on the line PQ, the segment PAB may be equal to a given space.

Let O be the centre of the circle PAB; draw BO, and make BR perpendicular to PQ. Put r = the radius PO or BO, y = PB, φ = the measure of the angle OPB to radius 1, n = the measure of two right-angles, and a^2 = the given space. Then, because the triangle POB is isosceles, the angle POB is $= n - 2\varphi$, therefore PB is $= r\ (n - 2\varphi)$, and the area of the sector POB $= \frac{1}{2}r^2 (n - 2\varphi)$. Also the area of the triangle POB is $=$ PO × BO × $\frac{1}{2}$ sin POB (or $\frac{1}{2}$ sin BOQ) $= \frac{1}{2}r^2 \sin 2\varphi$; theref. the area of the segment PAB is $= \frac{1}{2}r^2 \left\{ (n - 2\varphi) - \sin 2\varphi \right\} = a^2$. But by the property of the circle PB^2 = PR × $2r$, and by trigonometry PR = PB × cos φ; therefore $r = y^2 \div 2y \cos\varphi = y \div 2\cos\varphi$, and if this value of r be substituted in the above equation we have

$$\frac{y^2}{8\cos\varphi}(n - 2\varphi - \sin 2\varphi) = a^2,\ \text{or}\ y = \frac{2a\cos\varphi\sqrt{2}}{\sqrt{(n - 2\varphi - \sin 2\varphi)}},$$ the

equation of the curve required in terms of y and φ, and which is independent on the radius of the circle as it ought to be.

When φ is $= 0$, y is $= 2\sqrt{(2 \div n)}$; so that if PI be taken $= 2a\sqrt{(2 \div n)}$, the curve will pass through I.

If instead of the area PAB being a given space, the arch PB be of a given length, the problem may be resolved with the same ease. For if c be the length of the arch we have $r\ (n - \varphi) = c$;

and by the property of the circle $r = \frac{y}{2\cos\varphi}$, hence $\frac{y}{2\cos\varphi} \times$

$(n - 2\varphi) = c$, and $y = \frac{2c\cos\varphi}{n - 2\varphi}$ the equation of the curve

in this case.

These

These two examples are only particular cases of a more general problem, namely, "To determine the curve BB′ which shall cut an indefinite number of curves of the same species given by position in such a manner that similar functions (such as the arcs, areas, &c.) of these curves shall all have the same value." That is if x and y be co-ordinates of one of the given curves and p be the parameter, radius, &c. which is supposed to be constant for the same curve but to vary in passing from one curve to another: then $\int P\dot{x}$ = a constant quantity, P being a function of x and p. Or if the curve be defined by an equation between the polar ordinate y, and the arch φ, then $\int P\dot{\varphi}$ = a constant quantity, P being a function of p and φ.

Now when the expressions $\int P\dot{x}$ and $\int P\dot{\varphi}$ are integrable we can always eliminate p as we have done r in the preceding examples, but when they are not integrable the problem is more difficult to resolve, and another kind of process is necessary for eliminating p. The method of treating the two expressions being precisely the same, we shall confine our attention to the latter on account of the examples that follow. By taking the fluxion of $\int P\dot{\varphi}$ according to the method of differencing *de curva in curvam* we have

$$P\dot{\varphi} + \dot{p}\int\left(\dot{\varphi}\left(\frac{\dot{P}}{\dot{p}}\right)\right) = 0^{*}, \text{ or } P\dot{\varphi} + Q\dot{p} = 0 \quad \ldots\ldots\ldots\ldots\ a$$

Q being $= \int \dot{\varphi}\left(\frac{\dot{P}}{\dot{p}}\right)$. And here, when Q is integrable, or depends on $\int P\dot{\varphi}$, we can determine the value of p by integrating the fluxional equation $P\dot{\varphi} + Q\dot{p} = 0$, and then p may be eliminated as before.

To illustrate what has been said we shall resolve the preceding examples by this method.

When the area PAB is a constant quantity. Draw Pb indefinitely near to PB, and Ba perpendicular to Pb; then $Ba = y\dot{\varphi}$, the fluxion of the area PAB $= -\frac{1}{2}y^{2}\dot{\varphi} = -2r^{2}\cos^{2}\varphi\dot{\varphi}$,

---

* See Bossutt's Traites des Calcul. Tome Second, P. 62.

and

and $\int -2r^2\cos^2\varphi\dot\varphi = a^2$. Here $P = -2r^2\cos^2\varphi$, $p = r$, $(\frac{\dot P}{\dot p}) = -4r\cos^2\varphi$, $Q = \int -4r\cos^2\varphi\dot\varphi = \frac{2}{r}\int P\dot\varphi = \frac{2a^2}{r}$; and these values of P and Q being substituted in equation α we have $-2r^2\cos^2\varphi\,\dot\varphi + \frac{2a^2\dot r}{r} = 0$; or, dividing by r^2,

$$-2\cos^2\varphi\,\dot\varphi + \frac{2a^2\dot r}{r^3} = 0.$$

But $2\cos^2\varphi = 1 + \cos 2\varphi$, hence $-\dot\varphi - \cos 2\varphi\dot\varphi + \frac{2a^2\dot r}{r^3} = 0$, and taking the correct fluents, we get

$n - 2\varphi - \sin 2\varphi - \frac{2a^2}{r^2} = 0$, or $r^2 = \frac{2a^2}{n - 2\varphi - \sin 2\varphi}$.

But from the equation of the curve

$r^2 = \frac{y^2}{4\cos^2\varphi}$, hence $\frac{2a^2}{n-2\varphi-\sin 2\varphi} = \frac{y^2}{4\cos^2\varphi}$, and $y = \frac{2a\cos\varphi\sqrt{2}}{\sqrt{(n-2\varphi-\sin 2\varphi)}}$ as before.

Again when the arch PB is constant, we have the fluxion of the arch $= -2r\dot\varphi$, and $\int -2r\dot\varphi = c$. Here $P = -2r$, $p = r$, $(\frac{\dot P}{\dot p}) = -2$, $Q = -2\varphi$; and these values of P and Q being put in equation α, we have $-2r\dot\varphi - 2\varphi\dot r = 0$, and taking the correct fluent, $rn - 2r\varphi = c$, or $r = \frac{c}{n - 2\varphi}$.

And from the nature of the curve $r = \frac{y}{2\cos\varphi}$, hence $\frac{c}{n-2\varphi} = \frac{y}{2\cos\varphi}$, and $y = \frac{2c\cos\varphi}{n-2\varphi}$ as before.

SECOND SOLUTION, *by Mr.* J. H. SWALE.

Draw to the point P (fig. 24. pl. 1.) the indefinite tangent RPT; also draw all the diameters PL meeting the circles in L, L′, &c. In PL take PQ such, that the circle on the diameter PQ may equal twice the given space; then since PQ cuts from the circle PGQ a segment equal to the given space the curve will pass through Q.

Now the areas of the similar segments PAB, PGG; PA′B′, PG″G′, and so on, will be as the squares of their bases PB, PG; PB′, PG′ and so on. The curve in question will therefore be the locus of all the points B, found by drawing, from a given point P in the periphery of a given circle, any chord PG and taking $PB^2 : PG^2$ :: semi-circle PGQ : segment PG′G.

Let the diameter $PQ = r$, the measure of the angle $QPG = \beta$, the measure of two right-angles or the semi-circumference to radius 1, $= t$. Then the area of the semi-circle $= \frac{1}{8}tr^2$; and if I be the centre of the circle PGQ, the angle $GIQ = 2\beta$, the arch $GQ = \frac{1}{2}r \times 2\beta$, and the area of the sector $QIG = \frac{1}{4}r^2\beta$. Also the area of the triangle $PGI = \frac{1}{8}r^2 \sin 2\beta$; therefore the area of the segment $PG'G = \frac{1}{8}tr^2 - \frac{1}{8}r^2 \sin 2\beta - \frac{1}{4}r^2\beta$, and

$$PB^2 : PG^2 :: \tfrac{1}{8}tr^2 : \tfrac{1}{8}tr^2 - \tfrac{1}{8}r^2 \sin 2\beta - \tfrac{1}{4}r^2\beta, \text{ or}$$

$$PB : PG :: \sqrt{t} : \sqrt{(t - \sin 2\beta - 2\beta)}.$$

But by trigonometry $PG = r \cos \beta$, therefore

$$PB = \frac{r \cos \beta \sqrt{t}}{\sqrt{(t - 2\beta - \sin 2\beta)}},$$ which is the equation of the curve in terms of the angle QPG and the line PB.

XXII. QUESTION 262, *by* AMICUS.

If two points be taken on any semi-diameter of an ellipse, so that the rectangle of the segments between them and the centre may be equal to the square of the semi-diameter; and from these points straight lines be inflected to any point in the periphery of the ellipse meeting it again in two other points; the straight line joining these points will be parallel to the tangent at the vertex of the diameter. Required the demonstration?

SOLUTION, *by Mr.* J. T. M'DONELD.

Let two points P and I (fig. 25. pl. 1.) be taken on any diameter EF of an ellipse, or hyperbola, so that PC × CI may be equal to the square of the semi-diameter CE or CF, and from these points let two straight lines PQ and IQ be inflected to any point Q in the curve meeting it again in L and M, then the line LM will be parallel to the tangent at E, or F.

It is proved in the Solution to Quest. 236, in the Repository, that if tangents be drawn to the curve from P, the line AB, which joins the points of contact, will pass through I; also if PH be drawn parallel to AB meeting QI produced in H, and PQ meet AB in K, then

PQ : PL :: KQ : LK and HQ : HM :: IQ : IM.

But because PH is parallel to AB, PQ : KQ :: HQ : IQ; therefore PL : LK :: HM : IM, and consequently LM is parallel

parallel to AB; but AB is parallel to the tangent at E, as is well known; therefore LM is parallel to the tangent at E.

Cor. If LI be drawn to meet the curve in m, and HP in k; then Qm will be parallel to LM, and Pk = to PH.

Solutions were also received from Messrs. Bazley, Butterworth, Cavill, Filmatt, Kay, *and* Swale.

XXIII. QUESTION 263, *by Mr.* CUNLIFFE.

Having given the sum of the infinite series $\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\frac{x^4}{4^2}+$ &c. in any given circumstance of the value of x, it is required from thence to determine the sum of the infinite series $\frac{x}{2\cdot1^2}+\frac{x^2}{3\cdot2^2}+\frac{x^3}{4\cdot3^2}+\frac{x^4}{5\cdot4^2}+$ &c. in the same given circumstance of the value of x?

FIRST SOLUTION, *by* CANTAB. CAIENSIS.

Put $u=\frac{x}{1^2}+\frac{x^2}{2^2}+\frac{x^3}{3^2}+$ &c. and $s=$ the sum required.

Then $\frac{\dot{x}}{1-x}=\dot{x}+x\dot{x}+x^2\dot{x}+x^3\dot{x}+$ &c. by actual division

$$v \text{ or } \int\frac{\dot{x}}{1-x}=x+\frac{x^2}{2}+\frac{x^3}{3}+\text{\&c.}$$

$$\frac{v\dot{x}}{x}=\dot{u}=\dot{x}+\frac{x\dot{x}}{2}+\frac{x^2\dot{x}}{3}+\text{\&c.}$$

$$\int\frac{v\dot{x}}{x}=u=x+\frac{x^2}{2^2}+\frac{x^3}{3^2}+\text{\&c.}$$

$$\int u\dot{x}=\frac{x^2}{2\cdot1^2}+\frac{x^3}{3\cdot2^2}+\frac{x^4}{4\cdot3^2}+\text{\&c.}$$

$$\text{Therefore } s=\frac{1}{x}\int u\dot{x}.$$

But $\int u\dot{x}=ux-\int x\dot{u}=ux-\int v\dot{x}=ux-vx+\int\dot{x}v$,

and $\int x\dot{v}=\int\frac{x\dot{x}}{1-x}=\int\frac{\dot{x}}{1-x}-\int\dot{x}=v-x$, therefore

$$\int u\dot{x}=ux-vx-x+v, \text{ and}$$

$s=\frac{1}{x}\int u\dot{x}=u-1+\frac{(1-x)v}{x}$, where $v=$ the h.l. of $\frac{1}{1-x}$.

SECOND

SECOND SOLUTION, *by Mr.* BAZLEY.

Put $\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \&c. = S,$

and $\frac{x}{2 \cdot 1^2} + \frac{x^2}{3 \cdot 2^2} + \frac{x^3}{4 \cdot 3^2} + \frac{x^4}{5 \cdot 4^2} + \&c. = s;$

then by subtraction $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \&c. = S - s;$

but $\frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \&c. = \int \frac{\dot{x}}{1-x} = \text{hyp.log.}\, \frac{1}{1-x} = z,$

hence $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \&c. = \frac{1}{x} \int z\dot{x} = S - s;$ but

$\int z\dot{x} = zx - \int x\dot{z},$ and $\int x\dot{z} = \int \frac{x\dot{x}}{1-x} = -x + \int \frac{\dot{x}}{1-x} = z$

consequently $S - s = \frac{1}{x}(zx - z + x),$ and

$s = S - 1 + \frac{(1-x)z}{x} = S - 1 + \frac{1-x}{x} \times \text{hyp.log.}\, \frac{1}{1-x}.$

THIRD SOLUTION, *by Mr.* JONATHAN REEVES.

Let $\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \&c. = s.$

Put $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \&c. = v = \text{hyp. log.}\, \frac{1}{1-x};$

then multiplying by $x - 1$, we have

$-x + \frac{x^2}{2} + \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} + \&c. = v(x-1),$

or $\frac{x}{2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \frac{x^4}{4 \cdot 5} + \&c. = \frac{v(x-1)}{x} + 1;$ let this be subtracted from the first series and there remains

$\frac{x}{2 \cdot 1^2} + \frac{x^2}{3 \cdot 2^2} + \frac{x^3}{4 \cdot 3^2} + \&c. = s - 1 - \frac{x-1}{x} v,$ the sum required.

FOURTH SOLUTION, *by Mr.* W. WALLACE, *R.M.College.*

Let $P = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c.$

$Q = \frac{x}{2 \cdot 1^2} + \frac{x^2}{3 \cdot 2^2} + \frac{x^3}{4 \cdot 3^2} + \&c.$

Then

Then

$$x d\,\mathrm{P} = dx\,(x + \frac{x^2}{2} + \frac{x^3}{3} + \&c.),$$

$$d(x\mathrm{Q}) = dx\,(\frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \&c.),$$

therefore, remarking that $x + \frac{x^2}{2} + \frac{x^3}{3} + \&c. = -\log.(1-x)$,

$$x d\,\mathrm{P} = -dx \log.(1-x),\ d(x\mathrm{Q}) = \mathrm{P}\,dx, \text{ and adding,}$$

$$\mathrm{P}\,dx + x d\,\mathrm{P} = d(x\mathrm{Q}) - dx \log.(1-x).$$

But $\mathrm{P}dx + xd\mathrm{P} = d(\mathrm{P}x)$, and $dx \log.(1-x) = d\{x \log.(1-x)\} - dx + \frac{dx}{1-x}$, therefore,

$$d(\mathrm{P}x) = d(x\mathrm{Q} - d\,\{x \log.(1-x)\} + dx - \frac{dx}{1-x},$$

and taking the fluents,

$$\mathrm{P}x = x\mathrm{Q} - x \log.(1-x) + x + \log.(1-x)\,;$$

$$\text{Hence } \mathrm{Q} = \mathrm{P} - 1 - \frac{1-x}{x} \log.(1-x).$$

And when $x = 1$, $\mathrm{Q} = \mathrm{P} - 1$.

Solutions were also received from Messrs. Cavill, Filmatt, Harvey, *and* Watts.

XXIV. QUESTION 264, *by Mr.* W. WALLACE.

If two circles have the same centre, and an equilateral triangle be described about either of them, and from any point in the circumference of the other circle perpendiculars be drawn to the sides of the triangle; the straight lines which joins the bottoms of the perpendiculars shall form a triangle given in magnitude. Required the demonstration ?

SOLUTION, *by Mr.* W. WALLACE, *the Proposer.*

Let the perpendiculars be denoted by a, b, c. Now it is pretty generally known, and it is demonstrated in Dr. Stewart's *General Theorems*, (see the Original Series of the Mathematical Repository, Vol. I. p.p. 18 and 23) that both $a + b + c$ and $a^2 + b^2 + c^2$ are constant quantities: Therefore $(a + b + c)^2 = a^2 + b^2 + c^2 + 2\,(ab + ac + bc)$ is constant, and consequently $ab + ac + bc$ is constant; and putting s for the sine of 120°, $\frac{1}{2}sab + \frac{1}{2}sac + \frac{1}{2}sbc$ is a constant quantity. But $\frac{1}{2}sab$ expresses the area of of the triangle formed by the perpendiculars *a* and *b* and the line joining the points in which they meet the sides of the triangle; and in like manner $\frac{1}{2}sac$, and $\frac{1}{2}sbc$ express the areas of the triangles formed by *a*, *c* and *b*, *c* and the lines joining the points in which they meet the sides of the triangle; therefore the sum of these triangles, which make up the triangle having their bases for its sides, is a constant quantity.

This demonstration is adapted to the case in which the equilateral triangle is described about the outer circle; but by attending to the signs of the perpendiculars, and the positions of the triangles which they form, it will appear that the demonstration applies also to the case in which the equilateral triangle is described about the inner circle.

SECOND SOLUTION, *by Mr.* JOHN KAY.

Let perpendiculars PE, PF, and PH (fig. 26. pl. 1.) be drawn from any point P, in the periphery of the circle *abc*, to the sides of the equilateral triangle ABC and let the points E, F and H be joined: then the triangle EFH is of a constant magnitude. For let an equilateral triangle be described about the circle *abc*, so that its sides may be parallel to the sides of the other triangle, and let those sides meet the perpendiculars in the points *f*, *g* and D. and let the line *ab* which joins the points of contact meet PH in Q. Then because the circles about which the equilateral triangles are described have the same centre, the lines DH, E*f*, and *g*F are equal, and of a constant magnitude. Also because the sides of the triangles are bisected in the points *a*, *b*, the line QD is equal to half the perpendicular of the triangle described about the circle *abc*. But when lines are drawn from any point within an equilateral triangle perpendicular to the sides, their sum is equal to the perpendicular of the triangle, as is well known; hence $Pf + Pg + PD = 2DG$, and $Pf + Pg = DQ - PQ$. Also $PE + PF = DQ - PQ + 2DH = QH + DH - PQ$. Join *a*P, *b*P, *f*Q and *g*Q; then, because of the right-angles at *f* and Q, the points *a*, *f*, P and Q are in a circle; therefore the angle $PQf = Paf = Pbg$ (Eu. 32. III.). Also because of the right-angles at *g* and Q, the points *b*, *g*, P and Q are in a circle; therefore the angle $gQP = gbP = Pab$ (Eu. 32. III.): whence the triangles *f*QP, PQ*g* are similar, and we have $Pf : PQ :: PQ : Pg$, or $Pf.Pg = PQ^2$. Hence $PE.PF = (Pf + DH)(Pg + DH) = Pf.Pg + DH(Pf + Pg + DH) = PQ^2 + DH(QH - PQ)$.

Also $PH(PE + PF) = (HQ + PQ)(HQ + DH - PQ) = HQ^2 + DH(QH + PQ) - PQ^2$, and by addition $PE.PF + PH.PE + PH.PF = HQ^2 + 2DH.QH =$ a given space.

Draw Em and Fn perpendicular to HP produced, and Ev perpendicular to FP produced. Then because the angles EPF, FPH and EPH are the supplements of the equal angles A, B, C, the angles EPm, FPn, EPv are each equal to the angle of an equilateral triangle, hence Em = Ev = PE × $\sqrt{\frac{3}{4}}$, and Fn = PF × $\sqrt{\frac{3}{4}}$. Therefore the sum of the rectangles PF.Ev, PH.Fn, PH.Em, or twice the area of the triangle HEF is to the sum of the rectangles PE.PF, PF.PH, PF.PH as $\sqrt{\frac{3}{4}}$ to 1, or the area of the triangle HEF $= \frac{1}{2}\sqrt{\frac{3}{4}} \times$ HQ (HQ + 2DH) a given magnitude.

The demonstration will be the same when the perpendiculars are drawn from a point in the exterior circle.

Solutions were also received from Messrs. Butterworth, Harvey, M'Doneld, Swale, *and* Watts.

XXV. QUESTION 265, *by Mr.* LOWRY.

Let there be given, a rectilineal figure of any number of sides, and a space S: there may be found as many straight lines a, b, c, d, e, &c. as the figure has sides, such, that if from any point P whatever within the figure, perpendiculars PA, PB, PC, PD, &c. be drawn to the sides of the figure; the sum of the rectangles $a \times$ PA, $b \times$ PB, $c \times$ PC, $d \times$ PD, &c. shall be equal to the given space?

FIRST SOLUTION, *by Mr.* JONATHAN REEVES.

Let A be the area of the given figure, and $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. its sides. Let $S \div 2A = n$; then if a be taken $= n\alpha$, $b = n\beta$, $c = n\gamma$, $d = n\delta$, $e = n\varepsilon$. &c. they will be the lines required.

For it is manifest that $PA \times a + PB \times b + PC \times c +$ &c. will be $= (S \div A) \times \frac{1}{2} (PA \times \alpha + PB \times \beta + PC \times \gamma + PD \times \delta +$ &c.$) = (S \div A) \times A = S$, the given space.

COR. Since $PA \times \alpha + PB \times \beta + PC \times \gamma + PD \times \delta +$ &c. $= 2A$, we have $PA \times \frac{\alpha}{n} + PB \times \frac{\beta}{n} + PC \times \frac{\gamma}{n} + PD \times \frac{\delta}{n} +$ &c. $= \frac{2A}{n} =$ a given quantity, n being a given line.

Hence it is obvious that if lines be drawn from any point P within the figure to make angles whose sines are $\frac{n}{\alpha}, \frac{n}{\beta}, \frac{n}{\gamma}, \frac{n}{\delta}$, &c. with the sides $\alpha, \beta, \gamma, \delta$, &c. the sum of these lines will be equal to a given quantity, $\frac{2A}{n}$.

SECOND

SECOND SOLUTION, *by Mr.* J H. SWALE.

Let FGHIK &c. (fig. 27. p. 1.) be a given rectilineal figure, whose perimeter or sum of the sides FG, GH, HI, &c. $= P$, and area $= P \times Q$.

Let $P \times R = S$, the given space; make $2P \times Q : P \times R = 2Q : R = FG : a = GH : b = HI : c$, &c. so shall a, b, c, &c. thus determined, be the lines to be found.

From any point P within the given figure, demit upon the respective sides, the perpendiculars PA, PB, PC, &c. Then, since by the comp., $FG : a = 2Q : R = 2Q \times P : R \times P = 2Q \times P : S = FG \times PA : a \times PA = GH : b = GH \times PB : b \times PB = HI : c = HI \times PC : c \times PC$, &c. we shall have by compounding $FG \times PA + GH \times PB + HI \times PC +$ &c. $: a \times PA + b \times PB + c \times PC +$ &c. $= 2Q \times P : S$. But, the antecedents, in this general proportion, are obviously equal to each other; therefore the consequents $a \times PA + b \times PB + c \times PC +$ &c. and S, the given space, are also equal.

Messrs. Butterworth *and* Toplis *answered this question.*

XXVI. QUESTION 266, *by Mr.* LOWRY.

Let P be a given point in the axis of any curve AC*b*, and let a tangent to the curve at the point C, meet the axis in E: draw PF perpendicular to the tangent, and CB perpendicular to the axis, and join PC: Then if m and n be any given numbers, and the position of the point C be such, that the product of $PC^m \times CB^n$ is a maximum or a minimum, the rectangle FC.CE is equal to $\frac{n}{m} PC^2$. Required the demonstration?

FIRST SOLUTION, *by Mr.* J. WALLACE, *Edinburgh.*

Fig. 27. pl. 1. Put $PB = x$, $BC = y$, then $PC^2 = x^2 + y^2$, and since by the question, $PC^m \times CB^n$ is a maximum or minimum, $PC^2 \times CB^{\frac{2n}{m}}$ or $(x^2 + y^2) \times y^{\frac{2n}{m}}$ must be a maximum or minimum; hence differentiating and reducing we find

$$xy \frac{dx}{dy} + y^2 + \frac{n}{m}(x^2 + y^2) = 0.$$

And by substituting and observing that $y \frac{dx}{dy}$ is the expression for the subtangent we get $BC^2 - BP \times BE + \frac{n}{m} PC^2 = 0$, or $\frac{n}{m} PC^2 = BP \times BE - BC^2$. But if on EP as a diameter we describe a circle (which will evidently pass through F) and produce BC to meet it in G and H, then because $EB \times BP = BG^2$, it follows that $EB \times BP - BC^2 = BG^2 - BC^2 = GC \times CH = EC \times CF$, therefore $EC \times CF = \frac{n}{m} PC^2$.

SECOND SOLUTION, *by Mr.* LOWRY.

Draw P*c* (fig. 29. pl. 1.) indefinitely near to PC, and let C*p* and *ca* be drawn perpendicular to P*c* and CB. Put $PC = z$, $BC = y$, and C*c*, the fluxion of the arch C*b*, $= \dot{s}$.

Then by the question, $z^m \ y^n =$ a *maximum*, or a *minimum*.

By taking the fluxion, and dividing by $z^{m-1} \ y^{n-1}$, we get $my\dot{z} + nz\dot{y} = 0$, or $\frac{n}{m} z = -\frac{\dot{z}}{\dot{y}} y$; hence $\frac{n}{m} z^2 = -\frac{\dot{z}}{\dot{y}} yz$. But by similar triangles

$$Cc\,(-\dot{s}) : pC\,(\dot{z}) :: PC\,(z) : CF = -\frac{\dot{z}}{\dot{s}} z, \text{ and}$$

$$Ca\,(\dot{y}) : Cc\,(\dot{s}) :: CB\,(y) : CE = \frac{\dot{s}}{\dot{y}} y; \text{ hence}$$

$$CF.CE = -\frac{\dot{z}}{\dot{s}} z \times \frac{\dot{s}}{\dot{y}} y = -\frac{\dot{z}}{\dot{y}} yz = \frac{n}{m} z^2, \text{ \& theref. } CF.CE = \frac{n}{m} PC^2.$$

Several other theorems of a like nature may be deduced with equal facility. For example if $PC^m \times (\text{arch } Cb)^n$ be a *max.* or *min.*; then $\frac{n}{m} PC^2 = \text{arch } Cb \times CF$. For by taking the fluxion of $z^m s^n$, we get $\frac{n}{m} z = -\frac{\dot{z}}{\dot{s}} s$. But $-\frac{\dot{z}}{\dot{s}} = \frac{CF}{z}$; hence $\frac{n}{m} z = \frac{s \times CF}{z}$, or $\frac{n}{m} z^2 = s \times CF$.

Again if $PC^m \times Bb^n$ be a *max.* or *min.*; then $\frac{n}{m} PC^2 = \frac{Bb}{BE} \times CF \times CE$. For putting $x = Bb$, and taking the fluxion of $z^m \ x^n$, we get $\frac{n}{m} z = -\frac{\dot{z}}{\dot{x}} x$. But $-\frac{\dot{z}}{\dot{y}} = \frac{CF.CE}{yz}$ and $\frac{\dot{y}}{\dot{x}} = \frac{y}{BE}$; hence $-\frac{\dot{z}}{\dot{x}} = \frac{CF.CE}{z.BE}$, and $\frac{n}{m} z^2 = \frac{x}{BE} \times CF \times CE$.

Also if $PB^m \times (\text{arch } Cb)^n$ be a *max.* or *min.*; then $\frac{n}{m} PB \times CE = BE.Cb$. For putting $PB = v$, and taking the fluxion of

$v^m s^n$, we get $\frac{n}{m} v = -\frac{\dot{v}}{\dot{s}} s$. But $-\frac{\dot{v}}{\dot{s}} = \frac{\mathrm{BE}}{\mathrm{CE}}$; hence $\frac{n}{m} v \times \mathrm{CE} = \mathrm{BE} \times s$.

Likewise if $\mathrm{PB}^m \times (\text{area CB}b)^n$ be a *max.* or *min.*; then $\frac{n}{m}\mathrm{PB} \times \mathrm{BC} =$ the area CB*b*. For taking the fluxion of $v^m a^n$, we have $\frac{n}{m} v = -\frac{\dot{v}}{\dot{a}} a = \frac{\dot{v}}{y\dot{v}} a = \frac{a}{y}$; hence $\frac{n}{m} vy = a$.

Mr. Swale *sent an answer to this question.*

XXVII. QUESTION 267, *by Mr.* LOWRY.

To find a curve, such, that any straight line being drawn through a given point to meet the curve in two points, the tangents to these points shall always intersect each other in a straight line given by position?

SOLUTION, *by Mr.* LOWRY.

Let P (fig. 30, 31. pl. 1.) be the given point, and LM the straight line given by position; and let EKF be the curve required, such, that any straight line being drawn through P to meet the curve in E and F, the tangents EB, FB may intersect in the line LM. Take any line PA for the axis of the curve, and draw the ordinates EG, FH parallel to LM: Also draw BI parallel to AP meeting EG in I.

Put $\mathrm{AP} = a$, $\mathrm{PG} = x$, $\mathrm{GE} = y$, $\mathrm{HP} = x'$ and $\mathrm{FH} = y'$. Then $\dot{x} : \dot{y} :: \mathrm{BI} = x \pm a : \mathrm{EI} = \frac{\dot{y}}{\dot{x}}(x \pm a)$, and $\mathrm{AB} = \mathrm{GI} = y - \frac{\dot{y}}{\dot{x}}(x \pm a) = \frac{y\dot{x} - x\dot{y}}{\dot{x}} \mp \frac{a\dot{y}}{\dot{x}} = \frac{y^2}{\dot{x}} \times \left(\frac{x}{y}\right)^{\cdot} \mp \frac{a\dot{y}}{\dot{x}}$.

In the same manner we find $\mathrm{AB} = \frac{y'^2}{\dot{x}'} \times \left(\frac{x'}{y'}\right)^{\cdot} \mp \frac{a\dot{y}'}{\dot{x}'}$.

Hence $\frac{y^2}{\dot{x}} \times \left(\frac{x}{y}\right)^{\cdot} \mp \frac{a\dot{y}}{\dot{x}} = \frac{y'^2}{\dot{x}'} \times \left(\frac{x'}{y}\right)^{\cdot} \mp \frac{a\dot{y}'}{\dot{x}'}$.

Now since $\frac{x}{y}$ is $= \frac{x'}{y'}$, it is obvious that the above equation will be satisfied when AB is any function of $\frac{x}{y}$ or $\frac{x'}{y'}$; that is, we may assume

$$\frac{y^2}{\dot x} \times \left(\frac{x}{y}\right)^{\cdot} \mp \frac{a\dot y}{\dot x} = f\left(\frac{x}{y}\right),$$

$$\text{or, } \frac{y'^2}{\dot x'} \times \left(\frac{x'}{y'}\right)^{\cdot} \mp \frac{a\dot y'}{\dot x'} = f\left(\frac{x'}{y'}\right).$$

Taking the first of these equations, and putting $u = \frac{x}{y}$ and $U = f\left(\frac{x}{y}\right)$, a function of $\frac{x}{y}$, we have $\dot x = u\dot y + y\dot u$, and $\frac{y^2\dot u \mp a\dot y}{u\dot y + y\dot u} = U$, or $(y^2 - Uy)\,\dot u = (Uu \pm a)\,\dot y$. Putting $P = \frac{1}{Uu \pm a}$, and transposing, we have $\dot y + PUy\dot u = Py^2\dot u$.

To integrate this equation, put $y = -\frac{1}{z}$; then $\dot y = \frac{\dot z}{z^2}$. and by substitution, $\frac{\dot z}{z^2} - PU\frac{\dot u}{z} = P\frac{\dot u}{z^2}$; or $\dot z - z \times PU\dot u = P\dot u$.

Put $\frac{\dot v}{v} = -PU\dot u$, then $\dot z + z\frac{\dot v}{v} = P\dot u$, or $v\dot z + z\dot v = P\dot u \times v$, and integrating $zv = \int P\dot u \times v + C$, or

$$z = \frac{1}{v} \times \left(\int P\dot u \times v + C\right).$$

By assigning different forms to the function U, and integrating the expression $\int - PU\dot u$, in order to obtain v; we may find (from the above formula for z), innumerable curves having the property required by the question.

Taking for U the simplest function of $\frac{x}{y}$, viz. $n\frac{x}{y}$, we have $U = nu$, $P = \frac{1}{nu^2 \pm a}$, $\frac{\dot v}{v} = -PU\dot u = -\frac{nu\dot u}{nu^2 \pm a} = -\frac{u\dot u}{u^2 \pm b}$ (putting $a \div n = b$); and by integrating

$\text{l.}v = -\frac{1}{2}\text{l.}(u^2 \pm b)$, or $v = \frac{1}{\sqrt{(u^2 \pm b)}}$. Hence $\int P\dot u \times v = \int \frac{\dot u}{n(u^2 \pm b)^{\frac{3}{2}}} = \pm\frac{u}{nb\sqrt{(u^2 \pm b)}}$, & $z = -\frac{1}{y} = \sqrt{(u^2 \pm b)}.\left(\frac{\pm u}{nb\sqrt{(u^2 \pm b)}} + C\right)$;

or putting $C = \frac{1}{c}$, in order to make the expressions homogeneous, we have $-\frac{1}{y} = \pm \frac{u}{nb} + \frac{\sqrt{(u^2 \pm b)}}{c}$.

By putting $\frac{x}{y}$ for u, and a for nb, and reducing the equation, we get $y^2 + \frac{a^2 - c^2}{a^3} nx^2 \mp \frac{2nc^2}{a^2} x - \frac{c^2 n}{a} = 0$, for the equation of the curve required, which is evidently a conic section; viz. a parabola when a is $= c$, an ellipse when a is $> c$, and an hyperbola when a is $< c$.

When $y = 0$, $\frac{a^2 - c^2}{a^2} x^2 \mp \frac{2c^2}{a} x - c^2 = 0$, hence $c^2 = \frac{a^2 x^2}{x^2 \pm 2ax + a^2}$, and $c = \pm \frac{ax}{x \pm a}$; or if m $(= \mathrm{PK})$ be the value of x when $y = 0$, then $c = \pm \frac{am}{x \pm m}$.

Now when $a = c$, the equation of the curve is $y^2 \mp (2x - a)n = 0$, or $y^2 = (x \pm \frac{a}{2})\, 2n$. Also in this case $-\frac{m}{m \pm a} = 1$, or $2m = \pm a$; hence $2n$ is the parameter of the parabola, and PA is bisected by the curve in K.

Again when a is $> c$, if the distance CP, between the given point and the centre, be put $= e$, the semi-diameter OK $= t$, and the parameter to that diameter $= p$, we have $y^2 + \frac{p}{2t} x^2 - \frac{pe}{t} x + \frac{p}{2t}(e^2 - t^2) = 0$, for the equation of the ellipse, and by comparing it with the equation above, we find $e = \pm \frac{ac^2}{a^2 - c^2}$, $t = \frac{ca^2}{a^2 - c^2}$, and $p = \frac{2cn}{a}$; hence $\mathrm{AC} = \frac{ac^2}{a^2 - c^2} \pm a = \pm \frac{a^3}{a^2 - c^2}$, and $t^2 = \mathrm{CK}^2 = \pm \frac{ac^2}{a^2 - c^2} \times \pm \frac{a^3}{a^2 - c^2} =$ PC × CA.

When $p = 2t$, we have $\frac{2ca^2}{a^2 - c^2} = \frac{2cn}{a}$, or $c^2 = a^2 \times \left(\frac{n - a}{n}\right)$, and the equation becomes $y^2 + x^2 \mp 2(n - a)\, x -$

$a(n - a) = 0$, which answers to a circle whose radius is $\sqrt{(n - a)\, n}$.

When a is $< c$, the diameter and parameter of the hyperbola are found in the same manner as in the ellipse.

This question was answered by Messrs. Butterworth, Kay, *and* Swale.

XXVIII. QUESTION 268, *by Mr.* LOWRY.

Two points A, B, and an ellipse being given by position; a point may be found such, that if straight lines be drawn from the points A, B, to any point C in the periphery of the ellipse, meeting it again in D and E; and DP be drawn to meet the ellipse in F: The line joining the points E, F, shall have a given ratio to the diameter which is parallel to that line?

SOLUTION, *by Mr.* LOWRY, *the Proposer.*

The solution of this question will be greatly facilitated by means of the following propositions.

Prop. 1. Fig. 32. Pl. 1. Let A, B be two given points, and GCD an ellipse given by position; draw AG to touch the curve at G, and let R be the diameter which is parallel to AG, and V the diameter which is parallel to AB. Let AB be divided in I, so that $IA \times AB : AG^2 :: V^2 : R^2$; from the points AB let two straight lines be inflected to any point C in the curve, meeting it again in E and D: and draw EI meeting the curve in F: Then DF is parallel to AB. For if it is not parallel to AB, let DH be parallel to AB, and let it meet the curve in Q, the line EI in H, and the line AE in L. Then if S be put for the diameter which is parallel to AC, we have by the property of the ellipse.

$AG^2 : AE.AC :: R^2 : S^2$, and by hypothesis $AI.AB : AG^2 :: V^2 : R^2$;
therefore $AI.AB : AE.AC :: V^2 : S^2$; but by sim. triangles

$$AI.AB : AE.AC :: LE.LC : LH.LD,$$

$$\text{therefore } LE.LC : LH.HD :: V^2 : S^2,$$

and by the property of the ellipse $LE.LC : LQ.LD :: V^2 : S^2$;
therefore $LH.LD$ is $= LQ.LD$, or $LH = LQ$, which is impossible, except the points H and Q coincide, and then DQ coincides with DE, and the line DF is parallel to AB.

Prop. II. Fig. 33. Pl. 1. Let two straight lines AC and CB be drawn from any point C in an ellipse, parallel to two straight lines given by position, meeting the curve again in A and B;

and

and let the line joining the points A, B be bisected in Q; then the semi-diameter OQ*w* will be divided at Q in a given ratio.

For let lines be drawn from the points A, C and B, parallel to either axis of the ellipse, meeting the other axis XY in E, K and F, and the circle described on that axis in *a*, *c* and *b*, and join *ca*, *cb*. Then because *a*E : AE : : *c*K : CK : : the axis XY : the other axis, the lines *ca*, CA when produced will intersect in the axis XY produced. And it is shewn in the same way that the lines *cb*, CB, when produced will intersect in the axis XY. Therefore, since the triangle ICH is given in species, and *c*K has to CK a given ratio; it is obvious that the angle *acb* is given in magnitude. Draw *wv* parallel to *a*B meeting XY in G; bisect *ab* in P, and join P*v*, PO and PQ, and produce PQ to meet XY in S; then because AB is bisected in Q, and *ab* in P, the line PQ is parallel to *a*A, or to *vw*; therefore PQ : PS (: : A*a* : AE) : : *vw* : *v*G, therefore the points *v*, P, O are in a straight line; and consequently OP : P*v* : : QO : Q*w*; but since *acb* is a given angle, the radius O*v*, which bisects *ab*, is divided in a given ratio at P, therefore O*w* is also divided in a given ratio at Q.

This proposition suggests some curious properties of the conic sections.

Solution of the Question.

Let the tangent AG (fig. 34. pl. 1.) be drawn, and let R be the diameter which is parallel to AG, and V the diameter which is parallel to AB. Make the rectangle AI.AB : AG^2 : : V^2 : R^2; then will I be a given point. Through the centre O, draw IO, meeting the ellipse in K and M, and make the rectangle PO.OI = the square of the semi-diameter OK, then P is the point required.

That is, if straight lines be inflected from the points A, B to any point C in the ellipse, meeting it again in D and E, and DF be drawn through P, to meet the ellipse in F, the line EF will have a given ratio to the diameter to which it is parallel.

Let DI meet the ellipse in H, and join HE and HF, and let the diameter OQ be drawn to bisect EF in R; then because AI.IB : AG^2 : : V^2 : R^2, the line HE is parallel to AB, (Prop. I.), and because OP.OI is = OK^2, the line HF is parallel to the tangent at K (Question 262), and therefore is parallel to a straight line given by position; hence, from the point H in the ellipse, straight lines HE and HF are drawn parallel to two straight lines given by position, therefore the semi-diameter OQ is divided at R in a given ratio (Prop. II.), and consequently the diameter parallel to EF will have to EF a given ratio.

Cor. When AB is parallel to the tangent at K, the lines DF and HF will coincide, or the line DE will pass through P.

Mr. Swale *sent an answer to this question.*

XXIX. QUESTION 269, *by Mr.* W. WALLACE.

Is it possible in every case to divide each of two equal but dissimilar rectilineal figures, into the same number of triangles, such, that those which constitute the one figure are respectively identical with those which constitute the other?

SOLUTION, *by Mr.* LOWRY.

The possibility of dividing any two equal but dissimilar figures in the manner specified in the question, will be apparent from the solutions of the following problems:

Prob. I. To divide a given triangle into parts which shall form a rectangle. Fig. 35. Pl. 1.

Let BA be the longest side of the triangle, and draw HG to bisect the other two sides AE and BE in H and G. Make EF perpendicular to HG, and the triangle is divided by the lines HG and EF in the manner required.

Draw AC and BD perpendicular to AB, meeting HG produced in C and D; then, because AE and BE are bisected in H and G, the line CD is parallel to AB, and the figure ACDB is a rectangle. Also the triangle AHC is evidently equal to the triangle HEF, and the triangle BDG equal to the triangle FEG; therefore the parts which compose the triangle AEB are identical to the parts which compose the rectangle ACDB.

Prob. II. To divide a given rectangle into parts which shall form a rectangle of a given length. Fig. 36, 37, 38, 39, Pl. 1.

Let ABCD be the given rectangle, and from C, to the line AB, apply Cd = the given length, and make DI perpendicular to Cd. On Cd describe a rectangle $Cbad$, having its height (Cd, or da) = DI, and let the side ba meet AB (produced if necessary) in M. Then M may fall between A and B as in fig. 36, on BA produced beyond A, as in fig. 37 and 38, or on AB produced beyond B, as in fig. 39. In the first case the rectangle is divided in the manner required by the lines CL, DI, and KM. For $ID = da = Cb$, and the angle $DCI = BdL = IDL = aMd = AMK = bCK$; therefore, the triangles CID and Mad, IDL and bCK, BLd and AMK are respectively identical; therefore the rectangle $Cbad$ is composed of the same identical parts (1, 2, 3, 4) as the given rectangle ACDB.

In the second case, the point I may either fall on Cd, or on

Cd

Cd produced: When it falls on Cd, the rectangle is divided as required by the lines Cd and DI (fig. 37.) For since $DI = da$, and the angle $ICD = dMa$, the triangles CDI and Mda are identical; and the trapeziums $IDBd$ and $bCAM$, having all their angles respectively equal, and the side $DI = Cb$ and $DB = CA$, they are identical; therefore the rectangle $Cbad$ is composed of the same identical parts (1, 3, 4) as the rectangle ACDB. When I falls on the line Cd produced (fig. 38.) let DI meet AB in G, and make $an = IG$ and $Cr = dI$. Also draw nm parallel to AB, and rs parallel to DI; then, because $DI = da$, the triangles GBD and KCA, Mad and CID are identical. Also, because $IG = an = bK$, and $Cr = dI$, the triangles bMK, anm, rCs and dIG are identical; therefore if the two first of these be taken from the triangle Mad, and the two last from the triangle CID, there will remain the identical parts $bKdmn$ and $rsDGd$. Therefore the rectangle $Cbad$ is composed of the same identical parts (1, 2, 3, 4) as the rectangle ACDB.

In the third case, let ab meet DB in N, and take $Am = BN$ and $dR = MB$, and draw mn parallel to ab, and RG parallel to DB; then, because $DI = da = bC$, the triangles CDI and Mda. IDL and bCK, AKM and BLd are respectively identical. Also, because $Am = Bn$ and $dR = BM$, the triangles Amn, BNM and RdG are identical, and taking away those equal parts from the triangles AKM and BLd, there remains the identical parts $mKNBn$ and NLCRM. Therefore the rectangle $Cbad$ is composed of the same identical parts (1, 2, 3, 4, 5) as the given rectangle ACDB.

Cor. When $DI = Cd$, the figure $Cdab$ will be a square, and Cd will be a mean proportional between CD and CA; hence any rectangle may be divided into parts that will form a square, by taking Cd a mean proportional between the sides of the rectangle, and drawing the lines as in fig. 37 and 38.

It is obvious from hence, how any given triangle may be divided into parts that will form a square.

What has been done will enable us to divide any two equal but dissimilar rectilineal figures into parts which shall be respectively identical. For by the first problem any triangle may be divided into parts that will form a rectangle, and by the second problem any rectangle may be divided into parts that will form a rectangle of a given length; therefore any triangle may be divided into parts that will form a rectangle of a given length.— Now let A and B be two equal but dissimilar rectilineal figures of any number of sides whatever. Let those figures be divided into triangles by drawing diagonals, and let each of the triangles which compose the figure A be divided (by the preceeding problems) into parts that will form a rectangle whose length is any given line l: let those rectangles be joined by applying the sides

l to

l to each other; then the rectilineal figure A will be divided into parts that will form a rectangle whose side is l. Let each of the triangles which compose the figure B be divided, in a similar manner, into parts that will form a rectangle whose length is l; then, by joining those rectangles, the whole rectilineal figure B will be divided into parts which will form a rectangle whose length is l. By this means the equal rectilineal figures A and B will be divided into parts that will form two equal and similar rectangles P and Q. Those rectangles will, however, be composed of dissimilar parts, but those parts may be rendered identical by dividing the rectangle P in a manner similar to that in which the rectangle Q is divided, and the rectangle Q in a manner similar to that in which the rectangle P is divided: Then it is obvious that the rectangles P and Q (and consequently the rectilineal figures A and B) will be divided into parts that are respectively indentical. And if those parts which are not triangular be divided into triangles by drawing diagonals, the question will be completely resolved.

The Rev. Mr. Toplis *sent a Solution to this question.*

XXX. PRIZE QUESTION 270, *by* Z.

What must be the form of a billiard table, so that a ball being placed upon it at a given point, and struck in any direction whatever; it may, after striking the edge of the table twice, return to the point from whence it set off?

FIRST SOLUTION, *by* CANTAB. CAIENSIS.

With any axis major describe an ellipse having a focus A, (fig. 40. pl. 1.) and ED = CA being taken, D will be the other focus. Now let the ball be sent in any direction AP. Then by the property of the ellipse, the angle APZ = angle DPY; and therefore the ball reflected at P any point, necessarily passes through D the other focus. For the same reason at p, the angle Dpy = angle Apz, and the ball will be reflected to A again. Hence the curve required is an ellipse.

And thus the question was answered by Messrs. Butterworth, Buxton, Filmatt, Kay, M'Doneld, Swale, Toplis, *and* Tyratatos.

SECOND SOLUTION, *by* Z. *the Proposer.*

This question is, in effect, the same as one proposed in the Acta Eruditorum for September, 1745, relating to the reflection of light. It has been resolved by Euler, in the *Novi Commentarii*

tarii, tom. X, and subsequently, by M. Biot, in a Memoir presented to the National Institute. It would be easy to shew that an ellipse would satisfy the conditions of the question, but, in problems of this kind, the chief merit of the solution consists in determining the curve by a direct and general investigation. The solution that follows is nearly the same as Biot's, given by La Croix, in the 3d Vol. of his Traité du Cal. Dif. et du Cal. Int. Article mixed differences.

Let A (fig. 41. pl. 1.) be the fixed point, and $aMbM'$ the curve required, such, that if the ball be placed at A, and struck in any direction AM, it will be reflected from M to M′, and from M′ to A. Draw any line AB to represent the axis of the curve, and let tangents be drawn at M and M′ to meet the axis in T and F, and intersect in *t*. Also draw M′Q parallel to AB, meeting MP (perpendicular to AB), produced in Q. Then since the angles of incidence and reflection are equal, the conditions of the problem are contained in the two equations

$$\text{angle AMT} = \text{angle M}'\text{M}t,\ \text{angle AM}'t' = \text{angle MM}'\text{T}'.$$

Put $\text{AP} = x$, $\text{PM} = y$, $\text{AP}' = x'$, $\text{P}'\text{M}' = y'$, $\frac{dy}{dx} = p$, $\frac{dy'}{dx'} = p'$ and $\frac{\Delta y}{\Delta x} = \text{P}$. Then

$$\tan \text{MM}'\text{Q} = \frac{\text{MQ}}{\text{M}'\text{Q}} = \frac{\text{MP} + \text{PQ}}{\text{M}'\text{Q}} = \frac{\text{MP} + \text{P}'\text{M}'}{\text{M}'\text{Q}} = \frac{y - y'}{x' - x}$$

(regarding y' as negative) $= \frac{-\Delta y}{\Delta x} = -\text{P}$.

By observing that the angles MM′Q, MOT, and MOT′ are equal, and considering the exterior angles of the triangles OMT, AMT, we find

$$\tan \text{M}'\text{M}t = \tan(\text{PTM} + \text{MM}'\text{Q}) = \frac{p - \text{P}}{1 + p\text{P}},$$

$$\tan \text{AMT} = \tan(\text{PAM} - \text{PTM}) = \frac{\frac{y}{x} - p}{1 + \frac{y}{x}p} = \frac{y - px}{x + py}:$$

hence, from the first condition, angle AMT = angle M′M*t*, we have $\frac{y - px}{x + py} = \frac{p - \text{P}}{1 + p\text{P}}$ .... (1).

Also, from the relation of the angles of the triangles OM′T′, AM′T′; we find

$$\tan \text{MM}'\text{T}' = -\tan(\text{P}'\text{T}'\text{M}' + \text{MM}'\text{Q}) = -\frac{p' - \text{P}}{1 + p'\text{P}}$$

tan

$$\tan AM't' = \tan(P'AM' + P'T'M') = \frac{-\frac{y'}{x'} + p'}{1 + \frac{y'}{x'}p'} = -\frac{y' - p'x'}{x' + p'y'}.$$

And from the second condition, angle $AM't'$ = angle $MM'T'$, we have $\frac{y' - p\,'}{x + py} = \frac{p - P}{1 + p'P}$ …………… …….(2).

Hence from the equations (1) and (2) we deduce

$$P = \frac{y - 2px - p^2y}{p\,x - 2py - x} \text{ ……………… …………} (3)$$

$$P = \frac{y' - 2p'x - p^2y'}{p'^2x' - 2py' - x} \text{ ………………………} (4);$$

therefore $\frac{y - 2px - p^2y}{p^2x - 2py - x} = \frac{y' - 2p'x' - p'^2y'}{p'^2x' - 2p'y' - x'}$ ……… (5).

The result of which is, that the function $\frac{y - 2px - p^2y}{p^2x - 2py - x}$ does not change when x becomes x': This is the hypothesis on which we must integrate equation (3), which corresponds to $\frac{\Delta y}{\Delta x} = Const.$ This expression, being integrated by the method of finite differences, gives $y = x \times Const + \varphi\,(Const.)$: hence we have $y = x \times P + \varphi\,(P)$, or

$$y = x\left\{\frac{y - 2px - p^2y}{p^2x - 2py - x}\right\} + \varphi\left\{\frac{y - 2px - p^2y}{p^2x - 2py - x}\right\}.$$

From this expression, by assigning different forms to the function $\varphi\left\{\frac{y - 2px - p^2y}{p^2x - 2py - x}\right\}$ we may find innumerable curves that will answer the conditions of the question.

By assuming $\varphi\left\{\frac{y - 2px - p^2y}{p^2x - 2py - x}\right\} = 0$, we have $y = x\left\{\frac{y - 2px - p^2y}{p^2x - 2py - x}\right\}$, which, by integration, gives $x^2 + y^2 = c^2$, an equation to a circle.

Again by assuming $\varphi\left\{\frac{y - 2px - p^2y}{p^2x - 2py - x}\right\} = -2C$, we have $y = (x - 2C)\left\{\frac{y - 2px - p^2y}{p^2x - 2py - x}\right\}$, which, by integration, gives an equation to an ellipse or hyperbola.

For a further detail I shall refer the Reader to the Memoir already mentioned.

CANTAB. CAIENSIS *is requested to send to Mr.* GLENDINNING'S *for the Medal for solving the Prize Question.*

NOTICES.

I. DEATH OF MR. CAVENDISH.

The scientific world has to regret the loss of one of the most distinguished Philosophers of this or any other age; we mean HENRY CAVENDISH, Esq. who died near London, in March last, at the advanced age of 79.

Every one acquainted with the present state of Physical Science must know how much Mr. Cavendish has contributed to its improvement: Indeed it may be truly said that his discoveries and those of Dr. Black form the basis of almost the whole Science of Chemistry. Since his death, Mr. Davy, with that warmth of feeling which might be expected from a kindred spirit, has in his Lecture at the British Institution pronounced an elegant and just eulogium on his character. He said that "of all the philosophers of the present age Mr. Cavendish combined in the highest degree, a depth and extent of mathematical knowledge, with delicacy and precision in the methods of experimental research. It might be said of him what can perhaps hardly be said of any other person, that, whatever he has done has been perfect at the moment of its production. His processes were all of a finished nature; executed by the hand of a master, they required no correction, and although many of them were performed in the very infancy of Chemical Philosophy, yet their accuracy and their beauty have remained unimpaired, amidst the progress of discovery, and their merits have been illustrated by discussion, and exalted by time.......... "Since the days of Newton England has sustained no loss so great as that of Mr. Cavendish. But it is to be regretted less, since like his great predecessor he died full of years and glory."

II.—BRIEF NOTICES OF THE PRINCIPAL PAPERS RELATING TO MATHEMATICS, IN THE MEMOIRS OF THE FRENCH NATIONAL INSTITUTE.

Vol. I. Contains, 1. Of the Motions of the Celestial Bodies on their Centres of Gravity, by La Place.—2. New determination of the Orbit of Mercury, by J. La Lande.—3. On the Place of the Node of Saturn's Ring, in 1790, by Flaugergues.

Vol. II. contains, 1. Report of Le Gendre and La Grange, on Two Memoirs of Analysis, by Professor Burmann.—2. Report on the Project of a Metalic Thermometer, presented by Regnier.—3. Report on the Measure of the Meridian of France, and on the results which have been deduced to determine the Bases of a New Metrical System.—4. On the Secular Equations of the Motions of the Moon, of its Apogee, and of its Nodes, by La Place.—5. On the Means of converting continued Circular Motions into Alternate Rectilineal Motions, by Prony.—6. On such Portions of an Hemispherical Vault as can have their Solidity expressed by an Algebraic Formula, by Bossut.—7. Result of Experiments on the Quantity of Labour Men can perform under various circumstances, by Coulomb.—8. Five Memoirs relating to Occultations, Eclipses, and the Comet, discovered in the year 1798, by J. La Lande and Messier.

Vol. III. contains, 1. Report from La Grange and Bossut, concerning a Memoir by Callet on the Summation of certain Periodical Series.—2. Report of a Memoir by Biot, on the Integrals of Equations of Finite Differences, by La Place and Prony. —3. Observations on the Tides at Teneriffe.—4. On the Motion of the Orbits of the Satellites of Saturn and Uranus, by La Place.—5. Determination of the Forces which bring Needles to their Magnetic Meridan, by Coulomb.—6. On the Theory of the Moon, by La Place.—7. Experiments to determine the Cohesion of Fluids, and the Laws of their Resistance in very Slow Motions, by Coulomb.—8. On the Passage of Mercury over the Sun, on the 5th of May, 1799, by Delambre.

Vol. IV. contains, 1. On the Integration of Equations of Partial Differences; and on the Vibration of Surfaces, by Biot. — 2. Two Memoirs on the Solstitial Distances of the Sun from the Zenith; on the Secular Diminution of the Obliquity of the Ecliptic; and on the Summer Solstice, by M. Duc La Chapelle. — 3. Memoir relating to a Work, by Maignon; on the Reduction of the Apparent Distance of the Moon from the Sun or a Star to the True Distance, by means of a Chart.—4. New Method of Determining the Inclination of a Magnetic Needle, by Coulomb.

Vol. V. contains, 1. Report relative to Masts of Vessels, in which is introduced Interesting Information relative to Nautical Mechanics, by Leveque.—2. Notice respecting the Great Logarithmic and Trigonometrical Tables, by Prony.—3. Report relative to the Grand Trigonometrical Tables, by Delambre.— 4. Explanation of a Point of History respecting certain Trigonometrical Tables, by Prony.—5 On the Stereographic Projection, by Delambre.—6. Description of Circles or Rings of Different Colours observed round the Moon, by Messier.—7. Description of a New Compass, adapted to Determine the Direction and Absolute Declination of the Magnetic Needle, by D. Cassini.

Cassini.—8. On the Passage of Mercury over the Disc of the Sun, on the 5th of May, 1799, by Messier.—9. Seven Astronomical Memoirs, relating to the Zodiac, represented on the Church at Strasburgh; on the Appulse of the Moon and Mars; and on the Motion of Venus and Mercury, by J. La Lande.

VOL. VI. contains, 1. New Formula for Reducing the Apparent into the True Distances of the Moon from the Sun or Star, by Le Gendre.—2. An Observation of the 18th Passage of Mercury over the Sun, November 9, 1802, by La Lande. —3. On the Calculus of Partial Differences, and on the Attractions of Spheriods, by Biot.—4. On the Measurement of Heights by the Barometer, by Ramond.—5. Several Astronomical Memoirs, by Messier, Burckhardt, and Méchain, relating to Comets observed in 1793 and 1801; Eclipses; and to the Planet Pallas.

The Class of Mathematical and Natural Science of the National Institute have also published the First Volume of a Work entitled, "Mémoires présentés a l'Institut des Sciences et Arts par divers Savans etrangers." This Volume contains the following Papers relating to Mathematical Subjects, viz. 1. Observations to determine the Opposition of Mars and Jupiter, by Bouvard.—2. On the Orbits of some Ancient Comets, by Burckhardt.—3. On Micrometers, by the same.—4. On Polyhedrometry, by L'Huillier.—5. On Mixed Differences, by Biot. —6. Integration of the Equations of the Propagation of Sound, by Parseval.—7. On Curves of Double Curvature, by Lancret. —8. On the Application of the Calcul of Variations to Problems in Mechanics, by Ampere.—9. Integration of Two Important Equations that occur in the Motion of Fluids, by Parseval.—10. A Method of Summing, by Definite Integrals, the Series given by the Theorem of La Grange, by means of which he finds a value that satisfies an algebraic or transcendental equation, by the same.—11. On Series, and on the Integration of an Equation of Partial Linear Differences of the Second Order, by the same.—12. Solution of a Problem proposed by D'Alembert, relative to the conditions of the Equilibrium of a Flexible String fastened to its two ends, and passing through a groove cut in a body, which is supported by the string, by De Nieuport.— 13. On the General Equation of Regular Polygons, and on the Division of any Arc whatever into Equal Parts, by the same.

III. FOREIGN BOOKS LATELY IMPORTED.

Œuvres d'Archimède, traduites, littéralement, avec un Commentaire. Par F. Peyrard, Professeur de Mathematiques et d'Astronomie au Lycée Bonaparte; Suivies d'un Mémoire du Traducteur, sur un nouveau Miroir ardent, et d'un autre Mémoire de M. Delambre, sur l'Arithmetique des Grecs. Seconde Edit. 2 tom. 8vo. pp. 996. Paris 1808.

Essai sur la Théorie des Nombres. Par A. M. Legendre.—Seconde Edit. 4to. pp. 516.

Supplement (Second) au dixieme Livre du Traité de Mecanique Céleste sur l'Action Capillaire, 4to. pp. 78.

Supplement au Traité de Mecanique Céleste, Vol. III. 4to. pp 24.

Traité de Topographie D'Arpentage et de Nivellement: Par. L. Puissant. 4to. pp. 350.

IV. MATHEMATICAL WORKS LATELY PUBLISHED.

The Philosophical Transactions of the Royal Society of London, for 1809. The Mathematical Papers contained in this Volume are, 1. An Account of a Method of Dividing Astronomical and other Instruments, by Mr. Troughton.—2. On an Improvement in the manner of Dividing Astronomical Instruments, by Mr. Cavendish.—3. On a Method of Examining the Divisions of Astronomical Instruments, by the Rev. Wm. Lax.—4. Description of a Reflective Goniometer, by Dr. Wollaston.—5. Continuation of Experiments for Investigating the cause of Coloured Concentric Rings and other Appearances of a Similar Nature, by Dr. Herschell.—6. On the Attractions of Homogeneous Ellipsoids, by Mr. Ivory.

A Treatise on Plane and Spherical Trigonometry, by Robert Woodhouse, A. M. F. R. S. 8vo. pp. 199. Pr. 7s. 6d.

Elements of Geometry, Geometrical Analysis, and Plane Trigonometry, with an Appendix, Notes, and Illustrations, by John Leslie, Professor of Mathematics in the University of Edinburgh, 8vo. pp. 494. Pr. 12s.

A Treatise on Algebra, by John Mole. 12mo. pp. 304. Pr. 7s.

The System of the World, according to Sir Isaac Newton, and subsequent Mathematicians. Translated from the French of M. L. Place, by J. Pond, F. R. S. 2 vols. 8vo. pp. 706. Pr. 15.

The Doctrine of Life Annuities and Assurances, by Francis Bayly.

An Introduction to Arithmetic, being the first part of a general Course of Mathematics, with Notes, containing the Reason of every Rule, demonstrated from the most simple and evident Principles; together with some of the most useful Properties of Numbers. By John Bonnycastle, Professor of Mathematics at the Royal Military Academy, Woolwich, 8vo. Pr. 8s.

A System of Conic Sections, adapted to the Study of Natural Philosophy, by the Rev. D. M. Peacock, A. M. 8vo.

An Examination of M. La Place's Theory of Capillary Action, by T. K. 8vo. pp. 36.

Lectures on the Elements of Algebra, by the Rev. B. Bridge, A. M. Professor of Mathematics in the East India College, 8vo. Pr. 10s. 6d.

ARTICLE II.

Solutions to Questions proposed in Number IX.

I. QUESTION 271, *by Mr.* JOHN DAWES, *Birmingham.*

The hypothenuse of a right-angled triangle being given, also one side of a rectangle, inscribed upon the other two sides; to determine the triangle when that rectangle is a maximum?

FIRST SOLUTION, *by Mr.* WALLACE, *R. M. College.*

Instead of confining the problem to the particular case of a right-angled triangle, I shall give a solution, supposing it to be expressed more generally as follows.

Having given one side of a triangle, the opposite angle, and one of the sides of a parallelogram inscribed upon the other two sides; to determine the triangle when that parallelogram is a maximum.

Plate 2, fig. 42 (A). Let ABD be the triangle, AD its given side, and B the given angle. Let CFGB be the inscribed parallelogram, and FG its given side; then, when its area is a maximum, BG will also be a maximum.

Let us suppose the lines BD and FG, and consequently the point F, to be given by position, and let that extremity of the line AD which is upon BG be conceived to change its position a little by moving from D to *d*, while the line itself passes through the given point F; then it is pretty evident that when BG is a maximum, the other extremity A will be carried along the line AB; let *d* F *a* be the new position of the line, which is indefinitely near to the former; and let A*p*, D*q* be perpendicular to *ad*.

The indefinitely small triangles A*pa*, D*qd* have manifestly *ap* = *dq*; therefore A*p* : D*q* : : tan *a* : tan *d*, but A*p* : D*q* :: AF : FD, and tan *a* and tan *d* are evidently the same as tan FAB and tan FDB; therefore AF : FD : : tan FAB : tan FDB, or, drawing DH parallel to AB, and AH parallel to BD, AF : FD : : tan FDH : tan FAH. Hence it follows that a line drawn from H to F is perpendicular to AD.

Put $\varphi =$ angle ADB, $\alpha =$ given angle HDB, $b =$ given line FG, and $c =$ given line AD;

Then because $\sin\varphi : \sin\alpha :: \text{FG} : \text{FD}$ and $\sin\alpha : \sin\varphi :: \text{AD} : \text{AB} = \text{HD}$, therefore, $\text{FD} = \frac{\sin\alpha}{\sin\varphi} b$, and $\text{DH} = \frac{\sin\varphi}{\sin\alpha} c$; but $\text{FD} : \text{DH} :: \cos(\alpha - \varphi) : 1$, therefore $\cos(\alpha - \varphi) : 1 :: \frac{\sin\alpha}{\sin\varphi} b : \frac{\sin\varphi}{\sin\alpha} c$; and hence

$$\sin^2\varphi \cos(\alpha - \varphi) = \frac{b}{c} \sin^2\alpha.$$

It will be easy to find such values of φ as satisfy this equation, by the help of the trigonometrical tables, and the common method of approximation, and this is evidently the best mode of resolving it.

If we suppose α to be a right-angle, then $\cos(\alpha - \varphi) = \sin\varphi$, and thus the equation becomes

$$\sin^3\varphi = \frac{b}{c}.$$

But in this case $\sin\varphi = \frac{AB}{AD} = \frac{AB}{c}$, therefore we have $AB^3 = bc^2$, therefore AB is the first of two mean proportionals between the given lines AD and FG.

SECOND SOLUTION, *by* G. V.

Put the given hypothenuse $= d$, CB or EG (fig. 42. pl. 2.) the given side of the inscribed rectangle $= b$, and the angle $DAB = \varphi$. Then $BD = a\sin\varphi$, and $DG = b\tan\varphi$; therefore $BG = a\sin\varphi - b\tan\varphi$, and consequently

$$b\,(a\sin\varphi - b\tan\varphi) = \text{a maximum},$$

or, rejecting the constant factor b,

$$a\sin\varphi - b\tan\varphi = \text{a maximum}.$$

Therefore $\frac{d(a\sin\varphi)}{d\varphi} - \frac{d(b\tan\varphi)}{d\varphi} = 0$, that is, $a\cos\varphi = b\sec^2\varphi = \frac{b}{\cos^2\varphi}$; therefore $a\cos^3\varphi = b$; and $a^3\cos^3\varphi = a^2 b$; that is $AB^3 = AD^2 \times BC$.

THIRD SOLUTION, *by Mr.* JOHN HIGHMAN, *Totness.*

Let the hypothenuse of the right angled triangle ABC (fig. 42, pl. 2,) be $=$ to a; BG, one side of the parallelogram BE, $= b$, and $BC = x$: Then $AB = \surd(a^2 - x^2)$, and by similar triangles $x : \surd(a^2 - x^2) :: x - b : (\surd(a^2 - x^2)(x - b)) \div x =$ EG, the other side of the parallelogram: But by the question $\frac{(x - b) \times \surd(a^2 - x^2) \times b}{x}$ must be a maximum, and consequently its fluxion $\frac{b^2a^2\dot{x} - bx^3\dot{x}}{x^2\surd(a^2 - x^2)} = 0$; therefore $b^2a^2 = b\,x^3$, and $x = \sqrt[3]{(b\,a^2)}$, and $AB = \surd(a^2 - x^2) = \surd(a^2 - (ba^2)^{\frac{2}{3}})$.

Solutions were also received from Messrs. Harvey *and* Zeno.

II. QUESTION 172, *by Mr.* WALLACE, *R. M. College.*

Supposing a to denote any given whole number, find a general rule for determining all possible pairs of whole numbers x, y, which are such that the arch whose tangent is $\frac{1}{a}$ is the sum of two arches whose tangents are $\frac{1}{x}$ and $\frac{1}{y}$, the radius of the circle being unity?

SOLUTION, *by Mr.* WALLACE, *the Proposer.*

I shall render this question a little more general, by supposing that it is proposed to find all the possible values of whole numbers x and y, such, that the arch whose tangent is $\frac{n}{a}$ is the sum of the arches whose tangents are $\frac{n}{x}$ and $\frac{n}{y}$. Let the arch whose tangent is $\frac{n}{a}$ be denoted by $A\frac{n}{a}$, and let a similar notation be applied to the arches whose tangents are $\frac{n}{x}$ and $\frac{n}{y}$.

Then, because by hypothesis $A\frac{n}{a} = A\frac{n}{x} + A\frac{n}{y}$, by the arithmetic of sines, $\frac{n}{a} = \frac{\frac{n}{x} + \frac{n}{y}}{1 - \frac{n^2}{xy}} = \frac{n(x+y)}{xy - n^2}$; hence we have $xy - n^2 = ax + ay$, and $y = a + \frac{n^2 + a^2}{x - a}$; and as y must be a whole number, $x - a$ must be a divisor of $n^2 + a^2$; let p be any divisor of this quantity and q the quotient; that is let $pq = n^2 + a^2$, then $x - a = p$ and $x = a + p$, and since $y = a + \frac{pq}{p}$, therefore $y = a + q$. These values of x and y indicate sufficiently the general rule required, which may also be expressed in symbols as follows:

Let p and q be any two such numbers that $n^2 + a^2 = pq$, then

$$A\frac{n}{a} = A\frac{n}{a+p} + A\frac{n}{a+q}.$$

Ex. 1. Let $n = 1$, and $a = 1$, then $n^2 + a^2 = 2 = 1 \times 2$: In this case p and q can each have only one value, viz. $p = 1$, $q = 2$, therefore $A\,1 = A\,\frac{1}{2} + A\,\frac{1}{3}$. Hence it appears that an arch of 45° is the sum of two arches, namely, that which has its tangent equal to $\frac{1}{2}$ the radius, and that which has its tangent $\frac{1}{3}$ of the radius; and this is the only way in which an arch of 45° can be resolved into two arches, having their tangents exact parts of the radius.

Ex. 2. Let $n = 1$, $a = 2$, then $n^2 + a^2 = 5 = 1 \times 5$, therefore in this case also p and q have only one value, viz. $p = 1$, $q = 5$. Hence we get $A\,\frac{1}{2} = A\,\frac{1}{3} + A\,\frac{1}{7}$.

Ex. 3. Next let $n = 1$, $a = 3$, here $n^2 + a^2 = 10 = 1 \times 10 = 2 \times 5$. In this case p and q have each two values, and no more, viz. $p = 1$, $q = 10$, and $p = 2$, $q = 5$; therefore, $A\,\frac{1}{3} = A\,\frac{1}{4} + A\,\frac{1}{13}$; also $A\,\frac{1}{3} = A\,\frac{1}{5} + A\,\frac{1}{8}$.

Because $A\,1 = A\,\frac{1}{2} + A\,\frac{1}{3}$ and $A\,\frac{1}{2} = A\,\frac{1}{3} + A\,\frac{1}{7}$, therefore, $A\,1 = 2\,A\,\frac{1}{3} + A\,\frac{1}{7}$. Similarly, from the two values of $A\frac{1}{3}$, we get $A\,1 = 2\,A\,\frac{1}{4} + A\,\frac{1}{7} + 2\,A\,\frac{1}{13}$ and $A\,1 = 2\,A\,\frac{1}{5} + A\,\frac{1}{7} + 2\,A\,\frac{1}{8}$.

These formulæ, and others of the same kind, which may be found by the rule investigated above, afford, as is well known, the most simple means of any for the determination of the ratio of the diameter of a circle to its circumference. Among the formulæ, which consist of three terms, the following, viz. $A\,1 = 3\,A\,\frac{1}{7} + 2\,A\,\frac{1}{8} + 2\,A\,\frac{1}{18}$ is particularly simple, on account of the denominator 18 admitting of being resolved into factors. It is also, as far as I know, new.

Solutions were received from Messrs. Highman *and* Zeno.

III. QUESTION 273, *by Mr.* WALLACE.

Let T (1) denote the sum of the angles of any polygon, T (3) the sum of all the products that can be made by taking them by threes, T (5) the sum of all the products that can be made by taking them by fives, and so on; then, supposing radius to be unity,

$$T(1) - T(3) + T(5) - T(7) + \&c. = 0.$$

The signs being + and — alternately. Required the demonstration.

FIRST SOLUTION, *by Mr.* WALLACE, *the Proposer.*

Let α, α', α'', &c. denote any angles, and t, t', t'', &c. their tangents, and let

$$P = (1 + t\sqrt{-1})\,(1 + t'\sqrt{-1})\,(1 + t''\sqrt{-1})\,\&c.$$
$$Q = (1 - t\sqrt{-1})\,(1 - t'\sqrt{-1})\,(1 - t''\sqrt{-1})\,\&c.$$

Then by the arithmetic of sines,

$$\tan(\alpha + \alpha' + \alpha'' + \&c. \left(= \frac{P - Q}{\sqrt{-1}\,(P + Q)},\right)$$

hence, developing the functions P and Q, and putting T (1) for the sum of the tangents, T (2) for their products taken two by two, T(3) for their products taken three by three, and so on, we get

$$\tan(\alpha + \alpha' + \alpha'' + \&c.) = \frac{T(1) - T(3) + T(5) - T(7) + \&c.}{1 - T(2) + T(4) - T(6) + \&c.}$$

Now if α, α', α'', &c. be supposed to denote the angles of a polygon, their sum is always a multiple, of two right angles, therefore $\tan(\alpha + \alpha' + \alpha'' + \&c.) = 0$, so that in this case $\frac{T(1) - T(3) + T(5) - T(7) + \&c.}{1 - T(2) + T(4) - T(6) + \&c.} = 0$, and consequently $T(1) - T(3) + T(5) - T(7) + \&c. = 0$, as was to be demonstrated.

Cor. If we suppose that $\alpha + \alpha' + \alpha'' + \&c.$ = any odd number of right angles, then $\tan(\alpha + \alpha' + \alpha'' + \&c.)$ &c. = infinity, therefore in this case $1 - T(2) + T(4) - T(6) + \&c. = 0$.

Scholium. If we suppose the number of arcs to be three; and that $\alpha + \alpha' + \alpha''$ = any even number of right angles, then, from *the foregoing* demonstration,

$$\tan\alpha + \tan\alpha' + \tan\alpha'' - \tan\alpha\,\tan\alpha'\,\tan\alpha'' = 0,$$

from which it appears that the sum of the tangents of three arcs which make up any even number of right angles is equal to their continual product. A case of this pretty enough trignometrical theorem was given in the Ladies' Diary for 1797, but it has since been made the subject of a particular memoir in the London Philosophical Transactions for 1808.

Again, if we suppose the sum of any three arcs to be equal to any odd number of right angles, then we have $1 - (tt' + tt'' + t't'') = 0$. Hence, dividing by $tt't''$ we get $\frac{1}{t} + \frac{1}{t'} + \frac{1}{t''} = \frac{1}{tt't}$. That is, $\cot \alpha + \cot \alpha' + \cot \alpha'' = \cot \alpha \cot \alpha' \cot \alpha''$. Hence it appears that the sum of the cotangents of three arcs which make any odd number of right angles is equal to their continual product.

SECOND SOLUTION, *by Mr.* P. PHELAN.

The tangent of the sum of two arches a, a', being $= \frac{\tan a + \tan a'}{1 - \tan a \tan a'}$, if t, t', t'' &c. be put for the tangents of the respective angles a, a', a'', &c. we shall have

$$\tan(a + a') = \frac{t + t'}{t - tt'}$$

$$\tan(a + a' + a'') = \frac{\tan(a + a') + t''}{1 - \tan(a + a')t''} = \frac{t + t' + t'' - tt't''}{1 - (tt' + tt'' + t't'')}$$

$$\tan(a + a' + a'' + a''') = \frac{\tan(a + a' + a'') + t'''}{1 - \tan(a + a' + a'')t'''} =$$

$$\frac{t + t' + t'' + t''' - (tt't'' + tt't''' + tt''t''') + t't''t''',}{1 - (tt' + tt'' + tt''' + t't'' + t't''' + t''t''') + tt't''t'''}$$

And so on. Hence it is evident that the numerator of the expression for the tangent of the sum of all the angles is composed of the sum of all the tangents t, t', t'', t''', &c. and the sum of all the products that can be made by taking them by threes, by fives, by sevens, &c. and that the denominator is composed of the radius (1), and the sum of all the products that can be made by the tangents, taking them by twos, by fours, by sixes, &c. the signs being + and — alternately.

Hence, by adopting the notation in the question, we have

$$\tan(a + a' + a''' + a''' \text{ \&c.}) = \frac{T(1) - T(3) + T(5) - T(7) + \text{\&c.}}{1 - T(2) + T(4) - T(6) + \text{\&c.}}$$

But the sum of the angles of any polygon being equal to twice as many right angles as the polygon has sides, less four; the sum of all the angles of the polygon will be equal to an even number of right angles, and consequently the tangent of that sum $= 0$,

$$\text{hence } 0 = \frac{T(1) - T(3) + T(5) - T(7) + \text{\&c.}}{1 - T(2) + T(4) - T(6) + \text{\&c.}},$$

And therefore T (1) — T (3) + T (5) — T (7) + &c. = 0, as was to be demonstrated.

And thus the question was answered by Zeno.

IV. QUESTION 274, *by Mr.* WALLACE.

Let A and B be two given points in the diameter of a circle equally distant from the centre. In A B take a point P, so that P C × rad. = A C². Let C P meet the circumference in D. Draw A E, B E to any point in the circumference, and take the arc D Q double D E, and join P Q. Then shall A E × B E = P Q × rad. Required the demonstration?

SOLUTION, *by Mr.* WALLACE, *the Proposer.*

Fig. 43. pl. 2. Join E C, Q C: Let D C be denoted by r, AC by a, P C by b, and the angle D C E by φ; then D C Q = $2\ \varphi$. By the nature of the circle A E = $\sqrt{(a^2+r^2-2\,a\,r\cos.\varphi)}$, but by hypothesis, $a^2 = br$; therefore A E = $\sqrt{(r(b+r-2\,a\cos\varphi))}$. Similarly B E = $\sqrt{(r(b+r+2\,a\cos^2\varphi))}$. Therefore AE + BE = $r\sqrt{((b+r)^2-4\,a^2\cos^2\varphi)}$, that is, putting again $b\,r$ for a^2, A E × B E = $r\sqrt{(b^2+r^2-2\,b\,r\,(1-2\cos^2\varphi))}$: but $1-2\cos^2\varphi = \cos 2\,\varphi$, and $\sqrt{(b^2+r^2-2\,b\,r\cos 2\,\varphi)} = r \times$ C Q; therefore A E × B E = r × C Q, as was to be demonstrated.

Remark. From this proposition we may easily resolve either of the following problems. From two given points in the diameter of a circle equally distant from the centre to draw two lines to meet in the circumference, so that their sum or difference may be given. For if A E × B E be given, then it is manifest from the proposition that P Q is given. And because A E² + B E² = 2 A C² + 2 C D², therefore (A E ± B E)² = 2 A C² + 2 C D² ± 2 C D × C Q, hence if A E ± B E be given, it is evident that C Q will also be given.

SECOND SOLUTION, *by Mr.* LOWRY.

Join E C, Q C (fig. 43, pl. 2), and draw P H parallel to A E, meeting E C in H, and join H Q: then because P H is parallel to A E, P C : H C : : A C, E C; and by hypothesis P C : A C :: A C : E C; therefore H C = A C. And because in the triangles A C E, C H Q, A C is = to H C and C E to C Q, and the angle A C E = to the angle E C Q, therefore A E is = to H Q, and the angle C A E = to the angle B H Q. But the angle P H Q is the sum of the angles C H Q, C H P; that is, equal to the sum of the angles C A E, C E A, or the angle BCE; therefore the triangles P H Q, B C E, have the angles at H and C equal, and the sides about these angles proportional, *viz.* P H : H Q = A E : : CB = C H : C E; therefore those *triangles are equiangular*, and H Q = A E : P Q : : E C : B E, *or A E × B E* = P Q × E C. Q. E. D.

V. QUESTION 275, *by* G. V.

It is required to find the nature of other curves than conic sections, having the property which is proved in the solution of question 217, to belong to the ellipse?

FIRST SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let P (fig. 44, pl. 2,) be a given point, AB a straight line given by position, and AP perpendicular to AB. Required the nature of curves, such, that any straight line being drawn through P to meet the curve in E and E′, the rectangle of the perpendiculars drawn from these points to the line AB may be equal to the square of AP.

Draw EI and E′K perpendicular to AP produced; and put $a =$ AP, $z =$ PE, $z' =$ PE′, and $\varphi =$ the angle EPA. Then $\text{PI} = z \cos\varphi$ and $\text{PK} = z' \cos\varphi$; hence $\text{GE} = \text{AI} = z\cos\varphi - a$, $\text{E'H} = \text{AK} = z'\cos\varphi - a$, and $\text{GE} \times \text{HE'} = (z\cos\varphi - a)(z'\cos\varphi - a) = \text{AP}^2 = a^2$, or $zz'\cos\varphi - (z+z')a = 0$..(1)

But since EE′ meets the curve in two points only, the relation of z and φ will be expressed by an equation of the second degree, the roots of which are z and z'; namely an equation of the form $z^2 - 2\text{P}z + \text{R} = 0$, where P and R denote certain functions of φ, the relation of which is to be determined from the conditions of the problem. Now, by the nature of equations, 2P is = the sum of the roots z, z', and R = their rectangle; that is $2\text{P} = z + z'$, and $\text{R} = zz'$; therefore, by substitution in equation (1), we have $\text{R}\cos\varphi - 2\text{P}a = 0$, or $\text{R} = 2\text{P}a \div \cos\varphi$; hence the equation $z^2 - 2\text{P}z + \text{R} = 0$, becomes

$$z^2 - 2\text{P}z + \frac{2\text{P}a}{\cos\varphi} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

which is a general equation for curves having the property required, where P may be any odd function of $\cos\varphi$ and $\sin\varphi$.

If the relation between PI and IE be required, let $x =$ PI and $y =$ IE; then $\cos\varphi = x \div z$, and $z^2 - 2\text{P}z + 2\text{P}az \div x = 0$, hence $z = (x-a)\dfrac{2\text{P}}{x}$, or $x^2 + y^2 = (x-a)^2 \dfrac{4\text{P}^2}{x^2}$ .... (3) the general equation of the curves in this case.

If $\text{P} = r\cos\varphi = rx \div z$, then $x^2 + y^2 = (x-a)^2\, 4r^2 \div z^2$, or $x^2 + y^2 = (x-a)\,2r$, an equation to a circle whose radius is $\surd(r^2 - 2ar)$, and the distance of its centre from $\text{P} = r$.

If $\text{P} = \dfrac{c^2 r\cos\varphi}{c^2\cos^2\varphi \pm r^2\sin^2\varphi} = \dfrac{c^2 r\,\frac{x}{z}}{c^2\frac{x^2}{z^2} \pm r^2\frac{y^2}{z^2}}$; then $z = (x-a)\dfrac{2c^2rz}{c^2x^2 \pm r^2y^2}$;

or $c^2x^2 \pm r^2y^2 = (x-a)\,2c^2r$, an equation to an ellipse, or hyperbola, whose semi-axes are $\surd(r^2 - 2ar)$ and $c\sqrt{\dfrac{r-2a}{r}}$, and the distance of the centre from $\text{P} = r$.

If $P = m \sin \varphi + n \cos \varphi = \frac{my}{z} + \frac{nx}{z}$, then $z = \frac{(x-a) \times 2\,(my+nx)}{zx}$, or $zx = 2\,(x-a)(my+nx)$ and $(x^2+y^2)\,x = 2(x-a)\,(my+nx)$ an equation to a line of the third order.

By taking other values of P we may find innumerable curves that will answer the conditions of the question.

In deriving equation (2) from the general equation $z^2 - 2Pz + R = 0$, we eliminated R by means of the equation $R \cos \varphi - 2Pa = 0$; we might, however, have eliminated P, and then we should have obtained a different expression for the general equation of the curves; for since $P = R \cos \varphi \div 2a$, we have $z^2 - (R \div a)\, z \cos \varphi + R = 0$,

$$\text{or } x^2 + y^2 = R\left(\frac{x-a}{a}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

And, here, by assuming proper values of R, we shall obtain equations to the same curves as before.

For instance, if $R = n^2$, a constant quantity, then $x^2 + y^2 = \frac{n^2}{a}(x-a)$, an equation to a circle; $\frac{n}{2a} \surd\,(n^2 - 4a^2)$ being the radius, and $n^2 \div 2a$ the distance of the centre from P.

Again, if $R = \frac{n^2 c^2}{c^2 \cos^2 \varphi \pm r^2 \sin^2 \varphi} = \frac{n^2 c^2}{c^2 \frac{x^2}{z^2} \pm r^2 \frac{y^2}{z^2}} = \frac{n^2 c^2 (z^2 + x^2)}{c^2 x^2 \pm r^2 y^2}$, then

$x^2 + y^2 = \frac{n^2 c^2 (y^2 + x^2)}{c^2 x^2 \pm r^2 y^2} \left(\frac{x-a}{a}\right)$, or $c^2 x^2 \pm r^2 y^2 = \frac{n^2}{a} c^2 (x-a)$, an equation to an ellipse or hyperbola whose semi-axes are r and c (r being $= \frac{n}{2a} \surd\,(n^2 - 4a^2)$, and the distance of P from the centre $= \frac{n^2}{a}$.

In this solution we have supposed the point P to be without the curve; but the same reasoning will apply when it is within the curve, due regard being had to the changes of the signs.

If AL be taken = AP, and LD be drawn perpendicular to AK, then it is evident, from the solution to question 217, that if any line be drawn from P to meet the curve in E and E′ and LD in D, the line PE′ is harmonically divided in the points P, E, D and E′: Moreover the rectangle CE . CE′ is = to the square of PC.

SECOND SOLUTION, *by Mr.* W. WALLACE.

The object of the question is evidently to find curves GD*d* (fig. 45, pl. 2,) having these two properties:—1. Any straight line drawn from a given point A shall either not meet the curve at all, or shall meet it only in two points D, *d*, that are different from

from the point A. 2. If DE, *de*, be perpendiculars drawn from D, *d*, the points in which a line drawn through A meets the curve to BC a straight line given by position, the rectangle DE × *de* shall be equal to a given space.

Draw AFH perpendicular to BC, and draw DH, *dh*, perpendicular to AF. Put $AD = z$, the angle $A = \varphi$, and the given line $AF = a$.

Because z must have only two values, these may be considered as the roots of a quadratic equation $z^2 - Pz + Q = 0$, where P and Q denote certain functions of the angle φ, which have each only one value corresponding to a given value of φ: and as while the angle A increases from 0 to 360°, the line AD will have twice the same position, but will be in contrary directions, *viz.* when $A = \varphi$, and $A = \varphi + \pi$, (π being put for 180°) in order that the equation $z^2 - Pz + Q$ may give only two values of z, these must have the same magnitude, but with opposite signs, when $A = \varphi$, and when $A = \varphi + \pi$; for then the points in which the line AD cuts the curve, will be the same in both cases: Now both these conditions will be satisfied if P be any odd function of $\sin\varphi$, and $\cos\varphi$; and Q any even function of the same quantities: that is, putting c for $\cos\varphi$, and s for $\sin\varphi$, P must be a function of the form $\alpha c + \beta s + \gamma c^3 + \delta c^2 s + \varepsilon c s^2 + \zeta s^3 +$ &c. and Q a function of the form $\alpha' + \beta' c^4 + \gamma' s^2 + \delta' c^4 + \varepsilon' c^2 s^2 + \zeta' s^4 +$ &c.; for then when φ changes to $\varphi + \pi$, P changes to $-$ P, and P^2, and Q remain as before, and so the values of z corresponding to φ being $+\frac{1}{2}P + \surd(\frac{1}{4}P^2 - Q)$ and $+\frac{1}{2}P - \surd(\frac{1}{4}P^2 - Q)$ those corresponding to $\varphi + \pi$, will be $-\frac{1}{2}P + \surd(\frac{1}{4}P^2 - Q)$ and $-\frac{1}{2}P - \surd(\frac{1}{4}P^2 - Q$.

Let AD and A*d*, the two values of z in the equation $z^2 - Pz + Q = 0$, be denoted by z' and z'', and let b^2 be the space to which the rectangle DE × *de* is to be equal. Then because $DE = FH = z' \cos\varphi - a$, and $de = Fh = z'' \cos\varphi - a$, we have by the question, $(z' \cos\varphi - a)\ (z'' \cos\varphi - a) = b^2$, or $z' z'' \cos^2\varphi - (z' + z'')\, a \cos\varphi + a^2 - b^2 = 0$; but by the theory of equations $z' z'' = Q$, and $z' + z'' = P$, therefore, $Q \cos^2\varphi - aP \cos\varphi + a^2 - b^2 = 0$, and hence

$$Q = \frac{aP}{\cos\varphi} + \frac{b^2 - a^2}{\cos^2\varphi},$$

this value of Q is evidently an even function of $\cos\varphi$ and $\sin\varphi$, as it ought to be. Substituting now this value of Q, we get

$$z^2 \cos^2\varphi - P\,(z \cos\varphi + a) \cos\varphi + b^2 - a^2 = 0 \ \ldots\ldots\ldots (1)$$

a general equation for curves having the property required.

We may transform this equation into another which shall involve only the co-ordinates of the curve; for this purpose let us put $AH = x$, and $HD = y$; then $\cos\varphi = \frac{x}{z}$ and $\sin\varphi = \frac{y}{z}$, and P must now be an odd function, of $\frac{x}{z}$ and $\frac{y}{z}$. Let M be any rational homogeneous function of z and y, of $n+1$ dimensions, and N a rational and homogeneous function of n dimensions; then $\frac{M}{Z^{n+1}}$ and $\frac{N}{Z^n}$ will be functions of $\frac{x}{z}$ and $\frac{y}{z}$; and as the one will be an odd and the other an even function, the quotient of the one divided by the other, that is $\frac{Nz}{M}$ will be an odd function of the same quantities; we may therefore assume $P = \frac{Nz}{M}$; accordingly, substituting this value for P, and also $\frac{x}{z}$ for cos ϕ in the equation (1), it becomes, after reduction,

$$(x^2 + b^2 - a^2)\,M - x\,(x + a)\,N = 0 \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

which is another general equation for curves having the property required.

Let us take $M = x + \alpha y$, $N = A$. These functions give us

$$(x^2 + b^2 - a^2)\,(x + \alpha y) - Ax\,(x + a) = 0,$$

which is an equation to a line of the third order, having the property required. If we suppose $a = b$ it becomes simply

$$x^2 + \alpha xy - Ax - Aa = 0,$$

an equation to a hyperbola.

Again let us suppose

$$M = x^2 + \alpha xy + \beta y^2,\ N = Ax + By;$$

then we have

$$(x^2 + b^2 - a^2)\,(x^2 + \alpha xy + \beta y^2) - x\,(x + a)\,(Ax + By) = 0,$$

which is an equation to a line of the fourth order. If however we suppose $b = a$, it becomes

$$x\,(x^2 + \alpha xy + \beta y^2) - (x + a)\,(Ax + By) = 0$$

an equation to a line of the third order; and if we again suppose $B = 0$, this last becomes

$$x^2 + \alpha xy + \beta y^2 - Ax - Aa = 0,$$

which is an equation to a conic section.

VI. QUESTION 276, *by Mr.* WALLACE.

If from the focus of a parabola, a perpendicular be drawn to any tangent to the curve, and a circle be described on the focus as a centre to pass through the vertex, the meridional parts corresponding to the arc of the circle between the vertex, and perpendicular to the tangent, is equal to the excess of the parabolic arc, between the vertex and point of contact above that portion of the tangent which is intercepted between the same point and the perpendicular. Required the demonstration?

FIRST SOLUTION, *by Mr.* WALLACE, *the Proposer.*

Let F (fig. 46, pl. 2.) be the focus, V the vertex, VBE a circle described on F as a centre with the radius FV; VA a tangent to the curve at the vertex, CA a tangent to the curve at any point C, meeting VA in A. Join FA meeting the circle in B. Then, from the nature of the parabola, FAC will be a right angle. Produce the tangent CA so that CAD may be equal to the parabolic arc CV. Then AD = arc CV — tan CA. Let C′A′D′ be another position of the tangent CAD indefinitely near to the former. Draw FB′A′, and produce FA to meet C′D′ in a.

Let FV = 1, BV = φ, the enlarged meridian to arc $\varphi = m$; also let the parabolic arc CV be expressed by z, and the tangent CA by t; then AD $= z - t$. By similar figures FB : FA :: BB′ : A′a; but FB : FA :: $\cos\varphi : 1$, and BB′ : A′a :: $d\varphi : d(z - t)$; therefore $\cos\varphi : 1 :: d\varphi : d(z - t)$, and $d(z-t) = \frac{d\varphi}{\cos\varphi}$: but it is well known that $dm = \frac{d\varphi}{\cos\varphi}$; therefore $d(z - t) = dm$, and $z - t = m$, as was to be demonstrated.

SECOND SOLUTION, *by Mr.* LOWRY.

Let F (fig. 47, pl. 2,) be the focus, V the vertex, and EG a tangent to the parabola at any point E. Let FP be drawn perpendicular to EG, meeting the circle described with the centre F and radius FV in I; then, by the question, the meridional parts corresponding to the arch VI is equal to the arch VE—tan PE. Draw EH, IK perpendicular to the axis VD, and let EN be a normal to the curve at E. Put s = the arch VI, the enlarged meridian corresponding to that arch $= z$, VF $= a = \frac{1}{4}$ of the parameter of the parabola, KI $= u$, VH $= x$ and HE $= y$; then by the nature of the projection, $\dot{z} : \dot{s} ::$ rad. : cos VFI, or $\dot{z} = \frac{a\dot{s}}{\cos \text{VFI}} = \frac{a\dot{s}}{\sqrt{(a^2 - u^2)}}$; but $\dot{s} = \frac{a\dot{u}}{\sqrt{(a^2 - u^2)}}$, therefore $\dot{z} = \frac{a^2\dot{u}}{a^2 - u^2}$, and $z = \int \frac{a^2\dot{u}}{a^2 - u^2} = a \times \text{l.} \sqrt{\frac{a+u}{a-u}}$, the meridional parts corresponding to the arch VI.

Again, by the nature of the parabola, HN $= 2a$, and GV = VH; therefore GN $= 2(a + x)$, and GF $= a + x$; hence FN = GF $= \frac{1}{2}$ GN, and PE $= \frac{1}{2}$ GE. But GE $= \sqrt{(4x^2 + y^2)} = \frac{y}{2a}\sqrt{(y^2 + 4a^2)}$; therefore EP $= \frac{y}{4a}\sqrt{(y^2 + 4a^2)}$. Now the length of the parabolic arch VE is $= \frac{y}{4a}\sqrt{(y^2 + 4a^2)} +$

$a \times \text{l.}\,\dfrac{y + \surd\,(y^2 + 4a^2)}{a}$; therefore VE − PE $= a \times \text{l.}\,\dfrac{y + \surd\,(y^2 + 4a^2)}{a}$; but, by similar triangles FK $= \surd\,(a^2 - u^2)$: KI $= u$: : HN $= 2a$: HE $= y = \dfrac{2\,au}{\surd\,(a^4 - u^2)}$; and this value of y being substituted in the expression $a \times \text{l.}\,\dfrac{y + \surd\,(y^2 + 4a^2)}{a}$ it becomes $a \times \text{l.}\,\surd\dfrac{a+u}{a-u}$, the same as the expression for the meridional parts corresponding to the arch VI : hence $z =$ VE − PE, as was to be demonstrated.

Mr. Phelan *answered this question.*

VII. QUESTION 277, *by Mr.* Cunliffe, *R. M. College.*

Suppose 50 oz. avoird. of silver made into a bowl, the inside of which is in the form of a hemi-spheroid, the depth equal to the diameter at the top; how many pints of wine will the bowl contain if it is 15-100ths of an inch thick throughout?

Solution, *by Mr.* Cunliffe, *the Proposer.*

Let MDBO (fig. 48, pl. 2.) represent a section of the internal part of the bowl, by a plane passing through the axis OB. Suppose DF, at right angles to a tangent at D, meets OB in F: in FD produced let DP be the thickness of the bowl; then P will be a point in the curve, by whose revolution about the axis OB, the external surface of the bowl is formed. Draw DE, PQ perpendicular to OB. Put OB $= a$, OM $= \frac{1}{2}a$, OE $= v$, OQ $= x$, QP $= y$. By known properties of the ellipse. DE$^2 = \frac{1}{4}\,(a^2 - v^2)$, and FE $= \frac{1}{4}v$; therefore DF $= \frac{1}{4}\surd\,(4a^2 - 3v^2)$. By means of the parallels DE, QP,
DF : DP : : EF : EQ $=$ DP $\times$ EF $\div$ DF $= bv \div \surd\,(4a^2 - 3v^2)$;
whence OQ $= x =$ OE $+$ EQ $= v + bv \div \surd\,(4a^2 - 3v^2)$.
Again,

$$\text{FD} : \text{DE} :: \text{FP} : \text{QP} = \frac{\text{DE} \times \text{FP}}{\text{FD}} = y = \frac{(4b + \surd\,\overline{4a^2 - - 3v^2}) \times \surd\,(a^2 - v^2)}{2\surd\,(4a^2 - 3v^2)},$$

$$y^2 = \left(\frac{4b^2}{4a^2 - 3v^2} + \frac{2b}{\surd\,(4a^2 - 3v^2)} + \tfrac{1}{4}\right) \times (a^2 - v^2), \text{and } \dot{x} = \frac{4a^2 b\dot{v}}{(4a^2 - 3v^2)^{\frac{3}{2}}} + \dot{v}$$

$$\text{whence } y^2\dot{x} = \frac{4b^2\dot{v}\,(a^2 - v^2)}{4a^2 - 3v^2} + \frac{2b\dot{v}\,(a^2 - v^2)}{\surd\,(4a^2 - 3v^2)} + \tfrac{1}{4}\,\dot{v}\,(a^2 - v^2) +$$

$$+\frac{16a^2b^3\dot{v}(a^2-v^2)}{(4a^2-3v^2)^{\frac{5}{2}}}+\frac{8a^2b^3\dot{v}(a^2-v^2)}{(4a^2-3v^2)^2}+\frac{a^2b^2\dot{v}(a^2-v^2)}{(4a^2-3v^2)^{\frac{3}{2}}}=$$

$$\tfrac{1}{4}\dot{v}(a^2-v^2)+\frac{12b^2\dot{v}(2a^4-3a^2v^2+v^4)}{(4a^2-3v^2)^2}+$$

$$\frac{b\dot{v}(36a^6+16a^4b^2-87a^4v^2-16a^2b^2v^2+69a^2v^4-18v^6}{(4a^2-3v^2)^{\frac{5}{2}}}.$$

Whence taking the fluents

$$\int y^2\dot{x}=\tfrac{1}{4}v\left(a^2-\frac{v^2}{3}\right)+4b^2v(\tfrac{3}{4}a^2-v^2)\div(4a^2-3v^2)+\frac{ab^2\mathrm{L}}{4\sqrt{3}}$$

$$+\frac{3bv^4-bv^5\left(\dfrac{93a^2+40b^2}{12}\right)+bv(5a^4+4a^2b^2)}{(4a^2-3v^2)^{\frac{3}{2}}}+a^2b\mathrm{R}\sqrt{\tfrac{1}{3}},$$

where L denotes the hyperbolic logarithm $(2a+v\sqrt{3})\div(2a-v\sqrt{3})$, and R denotes the length of a circular arc radius 1, and sine $v\sqrt{3}\div 2a$. When $v=a$, the expression becomes

$$\frac{a^3}{6}+ab^2+\frac{ab^2}{4\sqrt{3}}\times\text{h.l.}(7+4\sqrt{3})+\frac{a^2b}{4}+\frac{2b^3}{3}+\tfrac{1}{3}a^2b\mathrm{R}\sqrt{\tfrac{1}{3}},$$

where A denotes the length of a circular arc radius 1, and sine $\frac{1}{2}\sqrt{3}$, or A denotes the length of an arc of 60°.

The foregoing expression being multiplied by $p=3{\cdot}14159265$ will give the content of the whole external figure, and deducting from thence $\frac{1}{6}a^3p$ the content of the internal space leaves

$$p\times\left\{ab^2+\frac{ab^2\sqrt{3}}{12}\times\text{h.l.}(7+4\sqrt{3})+\frac{2b^3}{3}+\frac{a^2b}{4}+\tfrac{1}{3}a^2b\mathrm{A}\sqrt{3}\right\}$$

for the content of the remainder, or the quantity of metal in the vessel. Now $\frac{1}{12}\sqrt{3}\times\text{h.l.}(7+4\sqrt{3})=.380173$, and $\frac{1}{3}\mathrm{A}\sqrt{3}={\cdot}6046$ nearly. By means of which the foregoing general expression for the quantity of metal in the vessel will become $p\times({\cdot}8546a^2b+1{\cdot}380173\,ab^2+\frac{2}{3}b^3)=50$ oz. of silver $=8{\cdot}2012339$ cubic inches: whence $8546a^2b+1{\cdot}080173\,ab^2+\frac{2}{3}b^3=2{\cdot}6105338$, and by taking $b={\cdot}15$, as in the question, the equation may be reduced to $a^2+a\times.242249=20{\cdot}3470146$ from whence the value of a is found $=4{\cdot}391268$ inches, and the content of the vessel from thence will be found $44{\cdot}3372$ cubic inches, or little more than a pint and a half of wine.

SECOND SOLUTION; *by* ZENO.

Let $x=$ the diameter at the top, $=$ also to the depth of the bowl; $c={\cdot}3=$ twice the thickness, $\pi=3{\cdot}14159265$, and $a=8{\cdot}2012339$ cubic inches $=50$ oz. of silver of standard fineness. Then (Hutton's Mensuration P. 324) $2x^3 . \frac{\pi}{12}$ = the content of the bowl, and $(2x+c)(x+c)^2 \times \frac{\pi}{12} =$ the content of the bowl and silver together: Hence $(2x+c) \times (x+c)^2 \times \frac{\pi}{12} - 2x^3 . \frac{\pi}{12} = a$, or $(2x^3+5cx^2+4c^2x+c^3-2x^3) \times \frac{\pi}{12} = a$; therefore $x^2 + \frac{4c}{5}x + \frac{c^2}{5} = \frac{12a}{5c\pi}$, and by completing the square, $x = -\frac{2c}{5} \pm \sqrt{(\frac{12a}{5c\pi} - \frac{c^2}{25})} = 4{\cdot}449547$; hence $2x^3 . \frac{\pi}{12} = 46{\cdot}126$ cubic inches $= 1\frac{3}{5}$ pints nearly.

VIII. QUESTION 278, *by* HERMODORUS.

Given the perspective projection (or representation) of a straight line, and of one point A therein, and the vanishing point B (and its distance) of that straight line; to find another point C therein, so that AC may have a given ratio to the original thereof?

SOLUTION, *by* HERMODORUS, *the Proposer.*

Let D (fig. 49, pl. 2,) be the intersection of the original of AB, and let D be given; from B draw BO = the given distance and parallel thereto D*a* meeting OA joined in *a*. Let C be the point required, and let OC joined meet D*a*, produced if need be in *c*. Then it appears by Dr. B. Taylor's perspective, (1719) Prob. 3, that this question is reduced to Loc. 6. Lib. 2. Sect. Rationis.

N. B. If the ratio which AC is required to have to *ac* be that of *m* to *n*, then it appears by the cor. to the said prob. of Taylor that $CB = \frac{m}{n} \times \frac{BD \times Da}{DA}$.

A Solution to this question was received from Mr. PHELAN.

IX. QUESTION 279, *by* G. V.

It is required to determine the nature of the curve which touches an infinite number of lines of a given kind, described upon a plane according to some determinate law: for example, we may suppose the lines to be the parabolas described by a projectile thrown from an engine, with a given velocity at every possible elevation in a given vertical plane.

First Solution, *by Mr.* Lowry.

Let $f(x, y, a) = 0$, be the general equation of the given curves; the expression $f(x, y, a) = 0$, denoting a given function of the co-ordinates, x, y, and an arbitrary quantity a, which is constant for each curve, but varies from one curve to another. Then we have to find the nature of the curve that will touch all the curves that can be formed from the general equation $f(x, y, a) = 0$, by giving succeessively all possible values to a. Now the point of contact being common to both curves, they will have the same co-ordinates x, y, at that point, and the same equation between these co-ordinates, but with this difference, that the arbitrary quantity a will be constant in the equation of the given curve, but variable in that of the tangential curve. Or, in the tangential curve, a must be a function of x, (y being also a function of x), so that the equation of the curve may be expressed in terms of the co-ordinates x and y, independent of the arbitrary quantity a. Now since the curves have a common tangent, the expression $y\frac{\dot{x}}{\dot{y}}$ for the subtangent must be the same when a is constant as when it is variable, or a function of x; and it is clear that this can only be the case when the fluxion of the equation $f(x, y, a) = 0$, taken relatively to a only, is equal to 0*. The fluxion being taken on this supposition, we shall have the value of a in terms of x, and this value being substituted in the general equation $f(x, y, a) = 0$, there will result an equation containing only x and y, and which will be the equation of the curve sought.

In the example in the question, let ADB (fig. 50, pl. 2,) be one of the parabolas described by the projectile when thrown with a velocity sufficient to carry it to the vertical height AS. Draw AB parallel to the horizon, and put AH $= x$, HD $= y$, AS $= h$, and the angle of elevation BAG $= a$. Then by Simpson's fluxions, page 235, $y = x\frac{\sin a}{\cos a} - \frac{x^2}{4h\cos^2 a}$, or

---

* For if $u = f(x, y, a) = 0$, then $(u)^{\cdot} = \frac{\dot{u}}{\dot{x}}\dot{x} + \frac{\dot{u}}{\dot{y}}\dot{y} = 0$, a being constant; and $(u)^{\cdot} = \frac{\dot{u}}{\dot{y}}\dot{x} + \frac{\dot{u}}{\dot{y}}\dot{y} + \frac{\dot{u}}{\dot{a}}\dot{a} = 0$, a being variable. Hence it is obvious that the values of $\frac{\dot{x}}{\dot{y}}$, deduced from these equations, cannot possibly be the same unless $\frac{\dot{u}}{\dot{a}}\dot{a}$ be $= 0$. See Woodhouse's Principles of Analytical Calculation, page 98.

$y - x\frac{\sin a}{\cos a} + \frac{x^2}{4h\cos^2 a} = 0$, the general equation of the curves described by the projectiles, and which corresponds to the general equation $f(x, y, a) = 0$. By taking successively for a all the different angles of elevation, an infinite number of parabolas will be formed, and we have to determine the nature of the curve that will touch all these parabolas. Taking the fluxion of the above equation, considering a only as variable,

we have $\frac{x\dot{a}}{\cos^2 a} - \frac{\dot{a}x^2 \cos a \sin a}{2h\cos^4 a} = 0$, or $1 = \frac{x \sin a}{2h \cos a}$;

hence $\frac{\sin a}{\cos a} = \frac{2h}{x}$, and $\cos^2 a = \frac{x^2}{4h^2 + x^2}$.

These values being substituted in the general equation of the curves, we have $y = 2h - \frac{4h^2 + x^2}{4h}$, or $y = \frac{4h^2 - x^2}{4h}$, an equation to a parabola, $4h$ being the parameter, S the vertex, and A the focus. This agrees with the solution to question 195, where this problem is also resolved, but by an indirect method.

For a second example, let $y^2 + (a - x)^2 - ah = 0$, be the general equation of the curve, which is that of a circle whose radius is $\sqrt{ab}$, and the distance of the centre C from a given point A (fig. 51, pl. 2.) in the axis $= a$, x being $=$ AD, and $y =$ BD. Then by giving all possible values to a, an infinite number of circles will be formed, and we have to find the nature of the curve that will touch all these circles. Taking the fluxion, considering a only as variable, we have $-2x\dot{a} + 2a\dot{a} - \dot{a}h = 0$, or $a = \frac{h + 2x}{2}$; and this value being substituted for a in the given equation, we have $y^2 - hx - \frac{h^2}{4} = 0$, an equation to a parabola, $2h$ being the parameter, and the distance of the vertex from A $= \frac{1}{4}h$.

For a third example, let it be required to find the nature of the curve that shall touch all the normals that can be drawn to a given curve.

Let the proposed curve be the parabola AD (fig. 52, pl. 2,) whose axis is AH and vertex A. Let the normal DI touch the curve at O, and draw HO and DP perpendicular to AH. Put AP $= u$, PD $= v$, AB $= a$, the parameter of the parabola $= m$, BH $= x$, and HO $= y$: then the subnormal PI $= \frac{1}{2}m$, and IH $=$ AH $-$ AI $= a + x - u - \frac{1}{2}m$; hence, by similar triangles, IH $(a + x - u - \frac{m}{2})$: HO (y) :: P I $(\frac{m}{2})$: PD $= v =$

$\frac{\frac{1}{2}my}{a + x - u - \frac{1}{2}m}$. But by the nature of the parabola, $v = \surd(mu)$, hence $\frac{\frac{1}{2}my}{a + x - u - \frac{1}{2}m} = \surd(mu)$, or $(a + x - u - \frac{m}{2}) \surd(mu) - \frac{1}{2}my = 0$. Take the fluxion of this equation, considering u only as variable, then we have $\dot{u} \surd(mu) + (a + x - u - \frac{m}{2}) \times \frac{m\dot{u}}{\surd(mu)} = 0$, and by reduction $u = \frac{a + x - \frac{1}{2}m}{3}$. This value being substituted for u in the above equation, we have

$$\tfrac{2}{3}\left(a - x - \frac{m}{2}\right) \surd\left\{\frac{m}{3}\left(a + x - \frac{m}{2}\right)\right\} - \tfrac{1}{2}my = 0$$

or, $\left(a - x - \frac{m}{2}\right)^3 = \frac{27}{16}my^2$; and when $a = \frac{m}{2}$, $x^3 = \frac{27}{16}my^2$, an equation to the semicubical parabola.

Again let the given curve be a cycloid, and AEB (fig. 53, pl. 2,) the generating circle. Let the lines be drawn as before, and let DP meet the circle in E. Join EB, and put AP $= u$, PD $= v$, arch HI $= z$, AB $= a$, BH $= x$ and HO $= y$. Then by the nature of the curve, DI is parallel to IB; therefore, by similar triangles, EP $= \surd(au - u^2)$: PB $= a - u$ :: DP $= v$: the subnormal PI $= \frac{v(au - u)}{\surd(au - u^2)} = v\surd\frac{a - u}{u}$. Also IH $=$ AI $-$ AH $= u + v\surd(\frac{a - u}{u}) - a - x$: OH $= y$ :: PI $= v\surd\frac{a - u}{u}$: PD $= v$; therefore $y\surd\frac{a - u}{u} = (u - a - x) + v\surd\frac{a - u}{u}$, or $v = y + (a + x - u)\surd\frac{u}{a - u}$. But by the nature of the curve, $v = z + \surd(au - u^2)$; therefore $z + \surd(au - u^2) = y + (a + x - u)\surd\frac{u}{a - u}$. Taking the fluxion of this equation, z and u only being considered variable, and putting for $\dot{z}$ its value $\frac{a\dot{u}}{2\surd(au - u^2)}$, and dividing by $\dot{u}$, we have

$$\frac{a - u}{\surd(au - u^2)} = -\surd\frac{u}{a - u}(a + x - u)\frac{\frac{1}{2}au}{(a - u)^2} \times \surd\frac{a - u}{u},$$

and, by reducing, $a - u = (a - x - u)\frac{1}{2}a$; hence $u = a - x$. And this value of u being substituted in the above equation, it becomes

$$z + \sqrt{(ax - x^2)} = y + 2x\sqrt{\frac{a-x}{x}};\text{ hence}$$

$z - \sqrt{(ax - x^2)} = y$, the equation of the curve required, which is that of a cycloid equal in all respects to the given one, but placed in an inverted position.

It is obvious that the curve required is the evolute, and the line DO the radius of curvature, of the given curve AD.

If it were required to find the curve which is the caustic of reflection or refraction for any proposed curve, the process would be nearly the same as in the preceding example. The 218th question of the Repository, (and indeed any other of the like kind when the nature of the curve is to be determined from certain properties of the tangent*) may be resolved in a similar manner. Thus if AB $= u$, (fig. to the solution to that question), AE $= x$, ED $= y$, and the sum of the sides $= a$; then by similar triangles, $u - x : y :: u : \text{AC} = \frac{uy}{u-y}$; hence

$$u + \frac{uy}{u-x} = a,\text{ or } u^2 - ux + uy - au + ax = 0.$$

Taking the fluxion, supposing u only to vary, and dividing by $\dot{u}$; then $2u - x + y = a$, or $u = \frac{a+x-y}{2}$; and this value being substituted for u in the above equation, we have, after proper reduction, $2ax = (a + x - y)^2$; an equation to the parabola. It is obvious that this process is precisely the same as that for deducing the particular integral from the general one in the solution to the question on page 214.

See also Lesson 17, " Lessons sur le Calcul des Functions," par M. La Grange.

SECOND SOLUTION, *by Mr.* WALLACE, *R. M. College.*

It is pretty evident that the curve whose nature is required must pass through the intersections of every two adjoining lines described as stated in the question, supposing these lines indefinitely near to one another. And this principle has suggested the following solution.

Let F (x, y, p) denote any expression which, besides constant quantities, involves two variable quantities x, y, and an indeterminate quantity p. Then, supposing x and y to be the co-ordinates of a curve, and p a quantity which may be called its *parameter*, because we are to suppose it analogous to the parameter of a conic section, or to the radius of a circle, &c. the nature of the lines described, as mentioned in the question, may be expressed generally by the equation

---

* Several examples of this kind may be found in Landen's Residual Analysis, Chap. VI. where they are resolved by a different method.

$$F(x, y, p) = 0,$$

the parameter p being supposed to have the same value for every point in any individual curve, but to have different values in different curves.

Now let ACE (fig. 54, pl. 2,) be any one of the lines, and let $AB = x$, and $BC = y$. Conceive its parameter p to change, and become $p + i$, and let A*ce* be the line which has $p + i$ for its parameter; and, supposing both lines ACE, A*ce* to have a common abscisa x, let $BC = y'$ be the new value of the ordinate, then the equation of the line A*ce* will be

$$F(x, y', p + i) = 0.$$

This expression by developement becomes

$$F(x, y, p) + \left(\frac{d\,F(x, y', p)}{dp}\right) i + Pi^2 + Qi^3 + \&c. = 0.$$

where P, Q, &c. are put to denote briefly certain functions derived from the expression $F(x, y, p)$. Now if c' be the point in which the two lines ACE, A*ce* intersect one another, it is evident that at this point $y' = y$, and therefore $F(x, y', p) = 0$. Hence it follows that if we suppose x and y to denote, not the co-ordinates generally, but only the co-ordinates at c', the intersection of the two curves, we shall have

$$\left(\frac{d\,F(x, y, p)}{dp}\right) i + Pi^2 + Qi^3 + \&c. = 0,$$

or, dividing by i,

$$\left(\frac{d\,F(x, y, p)}{dp}\right) + Pi + Qi^2 + \&c. = 0;$$

and this must be true, whatever be the magnitude of i. Suppose now i to be indefinitely little, then the lines ACE, A*ce* will approach indefinitely near to one another, and as in this case c', the intersection of the two lines, must be a point in the curve whose nature is required, as premised at the beginning of this solution, we may consider x and y as the co-ordinates of that curve.

Remarking now that the terms Pi, Qi^2, &c. all vanish, the relation of x and y will be evidently expressed by the equation

$$\left(\frac{d\,F(x, y, p)}{dp}\right) = 0.$$

This involves the indeterminate quantity p, but it may be exterminated by means of the equation

$$F(x, y, p) = 0,$$

and this done, the result will be an equation expressing the relation of x and y, the co-ordinates of the curve which touches all the lines ACE, as required.

I shall now apply the result of the preceding investigation to a few particular examples.

Example 1. Fig. 55, pl. 2. Let AE be a straight line given by position, and let an indefinite number of straight lines DC, &c. meet it in D, &c. so that perpendiculars DP, &c. drawn to them at their intersection with AE, shall pass through P, a given point: it is required to find the nature of the curve ACF that touches all these lines.

Let C be the point of contact of the curve and the straight line DC. Draw PA perpendicular to AE. Put PA $= a$, AB $= x$, BC $= y$, and AD $= p$. The triangles PAD, DBC are manifestly similar, therefore PA : AD :: DB : BC, that is, $a : p :: x - p : y$, therefore $ay = px - p^2$, and

$$F(x, y, p) = p^2 - px + ay = 0;$$

hence taking the fluxions, supposing p alone variable, we have

$$\left(\frac{d\,F(x, y, p)}{dp}\right) = 2p - x = 0;$$

therefore $p = \frac{1}{2}x$, and this value of p being substituted in the equation, it becomes $\frac{1}{4}x^2 - \frac{1}{2}x^2 + ay = 0$; hence we get $x^2 = 4ay$, from which it appears that the curve is a parabola having A for its vertex, AP for its axis, and 4AP for its principal parameter.

Example 2. Suppose an indefinite number of straight lines of a given length to terminate in two straight lines given by position at right angles to one another. Required the nature of a curve that shall touch all these lines.

Fig. 56, pl. 2. Let AB and AC be the two straight lines given by position, and DE any one of the indefinite lines of a given length; suppose F the point of contact, draw FG and FH perpendicular to AC and AB. Put AH $= x$, HF $= y$, the line DE $= a$: let the quantity which we have in the general investigation denoted by p be in this case the angle DEA. Then AD $= a \sin p$, and GD $= x \tan p$, therefore, the equation $F(x, y, p) = 0$, is here

$$y - a \sin p + x \tan p = 0;$$

and hence, taking the fluxions, p being variable, and x and y constant,

$$-a \cos p + \frac{x}{\cos^2 p} = 0.$$

Hence we find $x = a \cos^3 p$; and as AG must be related to the angle D, exactly as AH is to the angle E, it follows that $y = a \sin^3 p$: therefore $\cos^2 p = \left(\frac{x}{a}\right)^{\frac{2}{3}}$ and $\sin^2 p = \left(\frac{y}{a}\right)^{\frac{2}{3}}$, and $1 = \cos^2 p + \sin^2 p = \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}}$, so that the equation of the *curve is* $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Example

Ex. 3. Let us now suppose that the curve is required which shall touch all the parabolas that can be described by a projectile thrown from an engine with a given velocity in a given vertical plane. Let ACD (fig. 57. pl. 2.) be any one of these parabolas, HK its axis, AD an ordinate to its axis, AE $= a$ the the height due to the velocity of projection. Let AB $= x$, and BC $= y$ be the co-ordinates at C any point in the curve. Let the parameter of the axis be p, and considering AD as a function of the parameter, let it be denoted by q. By the theory of projectiles HK $= a - \frac{1}{4}p$, and by the nature of the parabola AH$^2 = p \times$ HK and AB $\times$ BD $= p \times$ BC; hence we have these two equations;

$$(1)\ qx - x^2 = py, \qquad (2)\ q^2 = 4ap - p^2.$$

Therefore, $F(x, y, p) = x^2 + py - qx = 0$,
and, taking the fluxions, supposing p and q to be the only variable quantities,

$$\left(\frac{dF(x, y, p)}{dp}\right) = y - \frac{xdq}{dp} = 0;$$

therefore $\frac{y}{x} = \frac{dq}{dp}$: Now from the first equation we find $\frac{y}{x} = \frac{q - x}{p}$; and from the second $\frac{dq}{dp} = \frac{2a - p}{q}$, consequently $\frac{q - x}{p} = \frac{2a - p}{q}$, and hence $q^2 - qx = 2ap - p^2$, or, substituting for q^2 its value in the second equation, $4ap - p^2 - qx = 2ap - p^2$; hence we get $qx = 2ap$, and from this, and the first equation, we find $p = \frac{x^2}{2a - y}$, $q = \frac{2ax}{2a - y}$: these values of p and q, being substituted in the second equation, and the common denominator rejected, it becomes

$$4a^2x^2 = (8a^2 - 4ay - x^2)\,x^2;$$

hence we get $4ay = 4a^2 - x^2$ for the equation of the curve; from which it appears to be a parabola having its focus at A the common intersection of all the parabolas; its axis perpendicular to the horizon; and its principal parameter equal to $4a$.

Mr. Phelan *sent an answer to this question.*

X. QUESTION 280, *by Mr.* CUNLIFFE.

Suppose one end of a string fastened to a point in the periphery of the base of a right cone whose convex surface is perfectly polished; and suppose the string to be wound three times tightly about the cone, and the other end thereof brought to a point in the same right line with the vertex and the first point. Required the dimensions of the cone, having given the length of the string and the distances of the two ends from the vertex.

SOLUTION

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

ANALYSIS. Let the sector ACA' (fig. 58, pl. 2.) represent the expanded surface of the cone; the points A, B the ends of the string; and AD, DE, EB the place or situation of the string upon the surface of the cone. It is known, from the principles of mechanics, that when the string is at rest stretched upon the smooth sides of the sector, then $\angle ADA' = \angle CDE$, and $\angle DEA = CEB$. In AD produced take DE' EB' respectively equal to DE, EB and join CE', CB'. Then in the triangles CDE', CDE; $DE' = DE$, by the construction, and $\angle ADA' = \angle CDE' = CDE$, by the nature of the problem; therefore the said triangles are equal in every respect, consequently $CE' = CE$, $\angle CE'B' = DEA$, and $DCE' = DCE$. Again in the triangles, $CE'B'$, CEB, $E'B' = EB$, $CE' = CE$ and $\angle CE'B' = DEA$ by construction and what has been deduced; and by the nature of the problem $\angle DEA = CEB$, therefore the said triangles are equal in all respects; consequently $CB' = CB$ and $\angle E'CB' = \angle ECB$; therefore $\angle DCE' = \angle DCE = \angle E'CB'$, and consequently $\angle ACB' = 3 \angle ACD$. Whence the following construction is derived.

Form the triangle ACB' having the side AB' equal to the given length of the string, the side AC equal to the given slant side of the cone, and the side CB' equal to the less distance of the end of the string from the vertex. Then with the centre C and radius CA describe the sector of a circle ACA' having $\angle ACA' = \frac{1}{3} \angle ACB$; then the arc AA' which is given, is equal to the periphery of the cone's base, from whence the diameter itself will be readily found.

It is manifest that the length of the string must not exceed the sum of the two distances of the ends from the vertex.

The solution would have been equally easy for any given number of rounds of the string: For example for 5 rounds, the $\angle ACA'$ must have been made equal to 1-5th of a given angle, &c.

Mr. Phelan *also sent an answer to this question.*

XI. QUESTION 281, *by* NUMERICUS.

To divide a given number into a proposed number of unequal parts (not less than 3), and also into the like number of other unequal parts, so that the sum of the squares of the latter parts shall be equal to the sum of the squares of the former.

For example; Let 150 be divided into 18 unequal numbers, and also into 18 other unequal numbers, so that the sum of the squares of the the latter 18 numbers shall be equal to the sum of the squares of the former 18.

SOLUTION, *by* NUMERICUS, *the Proposer.*

Lemma. Let n, a, b, c, &c. denote any numbers; and N a number to be determined: Then if $N + N - a + N - b + N - c$, &c. $= n + n + a + n + b + n + c$. &c. the sum of the squares will also be equal,

namely $N^2 + (N-a)^2 + (N-b)^2 + (N-c)^2$ &c. $= n^2 + (n+a)^2 + (n+b)^2 + (n+c)^2$ &c.

Suppose each series to be continued to t terms;

Then because $N + N - a + N - b + N - c +$ &c. $= n + n + a + n + b + n + c +$ &c.
we have $tN - a - b - c$, &c. $= tn + a + b + c$ &c. and

$N = \dfrac{tn + 2a + 2b + 2c.\ \&c.}{t}$; now if this be substituted for N in the equation $N^2 + (N-a)^2 + (n-b)^2 + (N-c)^2$ &c. $= n^2 + (n+a)^2 + (n+b)^2 + (n+c)^2$ &c. and the expressions actually squared, we get the result on one side of the equation exactly the same as the result on the other, the number of terms on each side being alike.

To apply this to the proposed question, we have $n + n + a + n + b + n + c$ &c. $= 150$; therefore let 150 be divided into 18 $(= t)$ unequal parts,

Suppose 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, 6, $6\frac{1}{2}$, 7, $7\frac{1}{2}$, 8, $8\frac{1}{2}$, 9, $9\frac{1}{2}$, 10, $10\frac{1}{2}$, 11, 12, $12\frac{1}{2}$, 13, (the sum of their squares being 1383):
then $n = 4$, and a, b, c, &c. will be the differences of 4 and $4\frac{1}{2}$, 4 and 5, 4 and $5\frac{1}{2}$ &c. successively,
or $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5, $5\frac{1}{2}$, 6, $6\frac{1}{2}$, 7, 8, $8\frac{1}{2}$, 9, and

$$N = \frac{18 \times 4 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 16 + 17 + 18}{18}$$

$= 12\frac{4}{6}$ and subtracting the differences $\frac{1}{2}$, 1, $1\frac{1}{2}$, &c. from $12\frac{4}{6}$ (or N) we have

$12\frac{4}{6}$ $12\frac{1}{6}$ $11\frac{4}{6}$ $11\frac{1}{6}$ $10\frac{4}{6}$ $10\frac{1}{6}$ $9\frac{4}{6}$ $9\frac{1}{6}$ $8\frac{4}{6}$ $8\frac{1}{6}$ $7\frac{4}{6}$ $7\frac{1}{6}$ $6\frac{4}{6}$ $6\frac{1}{6}$ $5\frac{4}{6}$ $4\frac{4}{6}$ $4\frac{1}{6}$ $3\frac{4}{6}$
the values of N, $N - a$, $N - b$, &c. or 18 other numbers whose sum $= 150$, and the sum of their squares $= 1383$.

Remark. If n, $n + a$, $n + b$, &c. are an arithmetical series, it follows from the nature of such progressions, that the value of N will be equal to the greatest term in that series; and N, $N - a$ $N - b$, &c. will give the series n, $n + a$, $n + b$, &c. but in a contrary order: In that case however, when the number of terms are not less than 6, we may proceed according to the following example, where it is required to divide the sum $1 + 2 + 3 + 4 + 5 + 6$ into 6 other numbers, so that the sum of their squares shall be $= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2$.

Let the 6 numbers be taken in two setts where the differences are unequal; suppose 1, 2, 5, and 3, 4, 6: Then $n = 1$, $a = 1$, $b = 4$, and $t = 3$ (in the first sett); and $N = \dfrac{tn + 2a + 2b}{t} = 4\frac{1}{3}$. from this take the differences 1 and 4, and we have $4\frac{1}{3}$, $3\frac{1}{3}$ and $\frac{1}{3}$ the

the three numbers whose sum $= 1 + 2 + 5$, and sum of their squares $= 1^2 + 2^2 + 5^2$. In the second sett $n = 3$, $a = 1$, $b = 3$, and $N = \frac{3 \times 3 + 2 + 6}{3} = 5\frac{2}{3}$, and $5\frac{2}{3}$, $4\frac{2}{3}$, $2\frac{2}{3}$ are three numbers whose sum $= 3 + 4 + 6$, and sum of their squares $= 3^2 + 4^2 + 6^2$: therefore $5\frac{2}{3}$, $4\frac{2}{3}$, $4\frac{1}{3}$, $3\frac{1}{3}$, $2\frac{2}{3}$, $\frac{1}{3}$ are 6 numbers answering the question.

It is evident that questions of this kind admit of a great variety of solutions. But answers may be obtained in a manner somewhat different from the above, by means of a *magic square containing magic squares*; for it is a property of such squares that the sum of the squares of the numbers in an outward rank or column, is always equal to the sum of the squares of the numbers in the opposite outward column; and as this is a property I have not found in any author, I shall subjoin a square of the kind.

The annexed square is a magic square containing magic squares: for if the 4 outward ranks are taken away, there will remain a magic square of 36 cells; and when the 4 outward ranks of that square are also omitted, a magic square of 16 cells is left.

6	62	8	60	55	7	61	1
56	16	23	48	47	46	15	9
2	22	25	39	38	28	43	63
54	21	36	30	31	33	44	11
12	45	32	34	35	29	20	53
52	41	37	27	26	40	24	13
14	50	42	17	18	19	49	51
64	3	57	5	10	58	4	59

Now if we take the numbers in any three sides of the three squares for one sett of 18 numbers; and the numbers in the three opposite sides for another sett of 18 numbers, and divide 150 into one sett, and also into another sett, proportional to those numbers, the two results will answer the question.

A Solution was received from Mr. Phelan.

XII. QUESTION 282, *by* L. N. M. C.

If a hexagon be described about a circle, or an ellipse, the three diagonals drawn through the opposite angular points of the figure will intersect each other in the same point. Required the demonstration?

First Solution, *by the Proposer.*

Prop. 1. Fig. 59, pl. 2. If four tangents aA, bA, dD, eD, be drawn to a circle or conic section, and AD meet the curve in m and p; then the lines ab, cd, which join the points of contact, and the tangents to the curve at *m* and *p*, when produced, intersect in the same point Q.

PROP. II. Fig. 60, pl. 2. If through any point P within a circle or conic section, chords *mp*, *nq*, *or*, &c. be drawn, the intersections of the tangents *m*Q, *p*Q; *n*R, *q*R, *o*S, *r*S, &c. will be in the same straight line. Or if the tangents intersect in a straight line, the lines *mp*, *nq*, *or*, which join the points of contact will pass through the same point P.

These propositions are so well known to geometricians that any demonstration of them here is unnecessary. See Hamilton's Conic Sections B. V. Prop. 5 and 8; and the Repository, Quest. 186.

Let ABCDEF (fig. 61, pl. 2.) be the circumscribing hexagon, and *a*, *b*, *c*, *d*, *e*, *f* the points of contact; and let the diagonals AD, BE, CF, be drawn to meet the curve in the points *m*, *p*; *n*, *q*; and *o*, *r*; then by Prop I. the chords *ab*, *ed*, and the tangents at *m* and *p*, when produced, meet in the same point Q; also the chords *bc*, *fe*, and the tangents at *n* and *q*, when produced, meet in the same point R; and the chords *cd*, *af*, and the tangents at *o* and *r*, when produced meet in the same point S. But when a hexagon as *abcdef* is inscribed in a circle, or conic section, the opposite sides when produced intersect in the same straight line (see Repository, quest. 180, and Simson's Conic Sections B.V.); therefore the points Q, R, S. or the intersection of the tangents at the extremities of the chords *mp*, *nq*, *or*, are in a straight line; therefore by Prop. II. these chords intersect in the same point; and consequently the diagonals AD, BE, and CF intersect in the same point.

A demonstration of this elegant proposition is given by M. Carnot in a small work published in 1806, entitled "Un Essai sur la Théorie des Transversales. But the discovery of the property he says is due to Brianchon.

SECOND SOLUTION, *by Mr.* P. PHELAN, *Carlow.*

Let *a*, *b*, *c*, *d*, *e*, *f* (fig. 62, pl. 2.) be the points where the hexagon ABCDEF touches the circle, and let the diagonals FC, AD, be drawn to intersect in P; then we have to prove that the points B, P, E are in a straight line. Let the lines joining the opposite points of contact be drawn so that the lines *ad*, *cf*, may intersect in H; the lines *ad*, *be* in G; and the lines *be*, *cf* in K; then it is a well known property of the circle, that the points H, G and K are in the diagonals FC, AD, and BE respectively, we have therefore only to prove that the points E, P, K are in a straight line, for then the diagonal EB will necessarily pass through P. For let *qn*, *mo* and P*p* be drawn through P parallel to *fc*, *ad*, and *bc* respectively; then by reason of the parallels,

cf.

qf : *f*F :: HP : H*a* :: *am* : *a*F; but *f*F is = *a*F, therefore *qf* is = *am*. And in the same manner it may be shewn that *pe* is = *do*. Again, if the tangents *f*F, CB be produced till they meet, they will be equal, and, therefore, because *qn* is parallel to *fc*, *qf* will be = *nc*. And, for the same reason, *am* will be = *do*, therefore the parts *am*, *cn*, *do*, *pe* and *fg* are equal; and since *e*E and *f*E are equal, *e*E : *f*E :: *pe* : *fq*; therefore (because *p*K and *f*K intersect in K, and *p*P and *q*P in P) the points E, P, K are in a straight line.

This method of demonstration will also apply to the ellipse, for it is easily shewn that the distances *am*, *cn*, *od*, *ep*, and *fg* are proportional to the diameters to which they are respectively parallel, and the tangents *e*E, *f*E being also proportional to the diameters to which they are parallel, *e*E will be to *f*E as *ep* to *fg*, and consequently the points E, P, K, B will be in a straight line. Indeed the demonstration for the ellipse may be easily inferred from that of the circle, by considering the ellipse as the section of a cylinder, or a cone, of which the circle is the base.

XIII. QUESTION 283, *by Mr.* CUNLIFFE.

To express the *n*th power of the arc of a circle in terms of its sine; radius being 1?

SOLUTION, *by Mr.* LOWRY.

Let a denote any arch, and y its sine; and put u for the series which is the expansion of a^n.

Then since $u = a^n$, we have $\dot{u} = na^{n-1}\dot{a}$; or because $\dot{a} = \frac{\dot{y}}{\sqrt{(1-y^2)}}$, and $a^{n-1} = \frac{u}{a}$, we have $\dot{u} = \frac{nu\dot{y}}{a\sqrt{(1-y^2)}}$, and $\frac{nu}{\sqrt{(1-y^2)}} = \frac{a\dot{u}}{\dot{y}}$.

Assume now $u = y^n + Ay^{n+2} + By^{n+4}\,Cy^{n+6} + \&c.$ then since, by the binomial theorem,

$$\frac{n}{\sqrt{(1-y^2)}} = n + \frac{n}{2}y^2 + \frac{1.3.n}{2.4}y^4 + \frac{1.3.5.n}{2.4.6}y^6 + \&c.$$

by multiplying these series we get

$$\frac{nu}{\sqrt{(1-y^2)}} = ny^2 + n(A + \tfrac{1}{2})y^{n+2} + n(B + \tfrac{1}{2}A + \tfrac{1.3}{2.4})y^{n+4}$$
$$+ n(C + \tfrac{1}{2}B + \tfrac{1.3}{2.4}A + \tfrac{1.3.5}{2.4.6})y^{n+6} + \&c.$$

Again taking the fluxion of the expression for u, and dividing by $\dot{y}$, we have

$\frac{\dot{u}}{\dot{y}} =$

$$\frac{\dot{u}}{\dot{y}} = ny^{n-1} + (n+2)Ay^{n+1} + (n+4)By^{n+3} + (n+6)Cy^{n+5} + \&c.$$

and since $a = \int \frac{\dot{y}}{\sqrt{(1-y^2)}} = y + \frac{1}{2.3}y^3 + \frac{1.3}{2.4.5}y^5 + \frac{1.3.5}{2.4.6.7}y^7 + \&c.$

by multiplying the two series we get

$$\frac{a\dot{u}}{\dot{y}} = ny^n + ((n+2)A + \frac{1.n}{2.3})y^{n+2} + ((n+4)B + \frac{1.(n+2)}{2.3}A + \frac{1.3.n}{2.4.5})y^{n+4}$$

$$+((n+6)C + \frac{1.(n+4)}{2.3}B + \frac{1.3.(n+2)}{2.4.5}A + \frac{1.3.5.n}{2.4.6.7})y^{n+6} + \&c.$$

And by equating the co-efficients of the like powers of y; in the two series which express the values of $\frac{n u}{\sqrt{(1-y^2)}}$ and $\frac{a\dot{u}}{\dot{y}}$, in order to make them identical, we have

$$A = \tfrac{1}{2}(\frac{1.2.n}{2.3})$$

$$B = \tfrac{1}{4}(\frac{1.3.4.n}{2.4.5} + \frac{2n-2}{2.3}A)$$

$$C = \tfrac{1}{6}(\frac{1.3.5.6.n}{2.4.6.7} + \frac{1.3.(4n-2)}{2.4.5}A + \frac{(2n-4)}{2.3}B)$$

$$D = \tfrac{1}{8}(\frac{1.3.5.7.8n}{2.4.6.8.9} + \frac{1.3.5(6n-2)}{2.4.6.7}A + \frac{1.3.(4n-4)}{2.4.5}B + \frac{(2n-6)}{2.3}C)$$

&c. &c. &c.

Where the law of continuation is obvious. And these values of A, B, C, D, &c. being substituted in the series assumed for u, we shall have the series for the nth power of a as required.

XIV. QUESTION 284, *by Mr.* CUNLIFFE.

Find three numbers whose sum shall be a square, and the sum of their squares an eighth power?

SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

I shall begin with the solution of the following problem, viz. To find three numbers whose sum shall be a square, and the sum of their squares a biquadrate?

Let x^2, $2xy$ and $2y^2$ denote the three numbers, then $x^2 + 2xy + 2y^2$ is to be a square, and $x^4 + 4x^2y^2 + 4y^4 = (x^2 + 2y^2)^2$ a biquadrate: that is $x^2 + 2xy + 2y^2$ and $x^2 + 2y^2$ must both be squares.

Put $x^2 + 2xy + 2y^2 = (ry - x)^2 = r^2y^2 - 2rxy + x^2$: whence $x = y \times (r^2 - 2) \div 2(r+1)$; take $y = 2(r+1)$; then $x = r^2 - 2$: whence $x^2 + 2y^2 = (r^2 - 2)^2 + 8(r+1)^2 =$ a square. The last expression will be a square when $r = -3$; but this value would give one of the numbers negative; wherefore in order to obtain positive numbers, take $r = v - 3$, then $r^2 - 2 = v^2 - 6v + 7$, and $r + 1 = v - 2$; whence $(r^2 - 2)^2 + 8(r + 1) = v^4 - 12v^3 + 58v^2 - 116v + 81$ = a square $= (v^2 - 6v - 9)^2 = v^4 - 12v^3 + 18v^2 + 108v + 81$; whence $v = \frac{224}{40} = \frac{28}{5}$; $r = v - 3 = \frac{13}{5}$, $x = r^2 - 2 = \frac{119}{25}$, and $y = 2(r + 1) = \frac{36}{5} = \frac{180}{25}$; therefore we may take $x = 119$, and $y = 180$: whence the three numbers will be $(119)^2$, $2 \times 119 \times 180$, and $2 \times (180)^2$. For their sum is $(349)^2$, and the sum of their squares $(281)^4$.

Now let each of the said numbers be multiplied by $(281)^2$, and they will become $(119)^2 \times (281)^2$, $2 \times 119 \times 180 \times (281)^2$, and $2 \times (180)^2 \times (281)^2$, which are three numbers whose sum is a square, viz. $(349)^2 \times (281)^2$, and the sum of their squares $(281)^8$. *Q. E. I.*

XV. QUESTION 284, *by* MECHANICUS.

If the ends of a perfectly flexible and inextensible string of a given length, be fastened at two given points on the surface of a sphere, and a small heavy body be at liberty to move freely along the string; required the position of the body when it is at rest on the surface of the sphere?

SOLUTION, *by* MECHANICUS, *the Proposer*.

Let A and B (fig. 63, pl. 2.) be the given points on the surface of the sphere, and C the position of the body when it is at rest. Then it is evident that the string will form two arches of great circles AC, BC. Let a vertical plane be drawn through C to meet the surface of the sphere in the great circle PIC, and let a plane be drawn through B to meet AC in D: then it is evident that the body will remain in the same position when the ends of the string ACD are fastened to the points A and B, as it would do if the ends of the string DBC were fastened to the points D, B. But when the points D, B, to which the ends of the thread are fastened, are in a horizontal line, the arch CI will evidently bisect the angle DCB when the body is at rest, and therefore the angle ACB is bisected by the vertical arch CI.

To find the distances AC, BC, let P be the highest point of the sphere, or the pole of the horizontal circle BD, and draw the arches AP, BP. Then the arch CI will pass through P, and the distances PA, PB will be given. Put $AC = x$, $BC = y$, $\angle ACI = BCI = z$, $\angle API = u$, $\angle BPI = v$, $AP = a$, $PB = b$, $AB = c$, and the length of the string $= l$. Then by spherical trigonometry we have in the triangle APC, $\sin PC = \dfrac{\sin^2 x - \sin^2 a}{\cos z \cos a \sin x - \sin a \cos x \cos u}$,

and in the triangle BPC, $\sin PC = \dfrac{\sin^2 y - \sin^2 b}{\cos z \cos b \sin y - \sin b \cos y \cos v}$;

hence

$$\frac{\sin^2 x - \sin^2 a}{\cos z \cos a \sin x - \sin a \cos x \cos u} = \frac{\sin^2 y - \sin^2 b}{\cos z \cos b \sin y - \sin b \cos y \cos v} \ldots (\alpha)$$

Again in the triangle ACB, we have $\cos ACB = (\cos c - \cos x \cos y) \div (\sin x \sin y)$. But since $\angle ACB = 2z$, $\cos ACB = 2 \cos^2 z - 1$, and therefore $\dfrac{\cos c - \cos x \cos y}{\sin x \sin y} = 2 \cos^2 z - 1$.

Now by means of this equation, and remarking that $\sin u = \dfrac{\sin x}{\sin a} \sin z$ and $\sin v = \dfrac{\sin y}{\sin b} \sin z$, we may eliminate z, u and v from equation (α); and then from the resulting equation, and the equation $x + y = l$, we may determine x and y.

XVI. QUESTION 286, *by* J. B.

Find the radius of a circle, such that the arch corresponding to a given versed sine may be a minimum.

SOLUTION, *by Mr.* LOWRY.

Let AB (fig. 64, pl. 2.) be the given versed sine, O the centre of the circle when the arch BEP is a minimum, and draw OP, PB. Let *opeb* be another sector of a circle similar to PEBO, and having 1 for the radius; draw *pa* perpendicular to *bo*, and put $z =$ the arch PB, $a =$ AB, $u =$ the arch *peb*, and $x = ab$; then by similar figures, $z : u :: a : x$; or $z = a \times \dfrac{u}{x}$, a minimum. Taking the fluxion of the variable part of this expression, viz. $\dfrac{u}{x}$, we have $\dot{u}x - u\dot{x} = 0$, or $x = u \times \dfrac{\dot{x}}{\dot{u}}$. But $x = 1 - \cos u$, and $\dot{x} = \dot{u} \sin u$; therefore, by substitution, $\dfrac{\dot{x}}{\dot{u}} = \sin u$, and $x = u \sin u$; or the versed sine *ba* is equal to the arch *peb*, multiplied by the sine *ap*; therefore AB × rad. OB = arch BEP × AP.

The question may be otherwise resolved as follows. Let $r =$ the radius BO, and $v =$ AO, then, by the property of the circle, $\dot{z} = \int \dfrac{r\dot{v}}{\sqrt{(r^2 - v^2)}}$, which is to be a minimum, and consequently its fluxion, taken on the supposiſion that r and v both vary, must be 0. Therefore taking the fluxion according to the method of differencing de curva in curvam, we have $\dfrac{r\dot{v}}{\sqrt{(r^2 - v^2)}} -$

$\dot{r}\int\frac{v^2\dot{v}}{(r^2-v^2)^{\frac{3}{2}}}=0$. But $\int\frac{v^2\dot{v}}{(r^2-v^2)^{\frac{3}{2}}}=\frac{v}{\sqrt{(r^2-v^2)}}-\int\frac{\dot{v}}{\sqrt{(r^2-v^2)}}$ $=\frac{v}{\sqrt{(r^2-v^2)}}-\frac{z}{r}$; hence $\frac{r\dot{v}}{\sqrt{(r^2-v^2)}}-\dot{r}\left(\frac{v}{\sqrt{(r^2-v^2)}}-\frac{z}{r}\right)$ $=0$. Also $r=a-v$, and $\dot{r}=-\dot{v}$; therefore $\frac{r}{\sqrt{(r^2-v^2)}}+$ $\frac{v}{\sqrt{r^2-v^2)}}-\frac{z}{r}=0$; or by reduction, $r(r+v)=z\sqrt{(r^2-v^2)}$, the same as before.

The same result may be obtained by the following method.

Put $\varphi=$ the angle PBA, $n=$ a semicircle, or the measure of two right angles, radius $=1$, and $y=$ PB: Then it appears from the 2nd Example, on pa. 28 of this vol. that $z=\frac{y(n-2\varphi)}{2\cos\varphi}$, and because $y=\frac{a}{\cos\varphi}$, we have $z=\frac{a(n-2\varphi)}{2\cos^2\varphi}$, a minimum, or $\frac{n}{\cos^2\varphi}-\frac{2\varphi}{\cos^2\varphi}=$ a minimum. Therefore, taking fluxions,

$$-n(\cos^2\varphi)^{\cdot}-2\dot{\varphi}\cos^2\varphi+2\varphi(\cos^2\varphi)^{\cdot}=0.$$

$$\text{or } (-n+2\varphi)(\cos^2\varphi)^{\cdot}-2\dot{\varphi}\cos^2\varphi=0.$$

But $(\cos^2\varphi)^{\cdot}=-2\varphi\sin\varphi\cos\varphi$; therefore $(n-2\varphi)\sin\varphi\cos\varphi-\cos^2\varphi=0$, or $(n-2\varphi)=\frac{\cos\varphi}{\sin\varphi}=\frac{AB}{AP}$, and since $n-2\varphi=$ the measure of the angle BOP, $=\frac{\text{arch BP}}{\text{rad. PO}}$, we have AB × PO = arch BP × AP, as before.

To find the arch whose versed sine is equal to the arch multiplied by the sine, radius being 1, it is easily found by a few trials that the arch is greater than 133° and less than 134°. Hence by proceeding as in the solution to question 255, we have

Log. 133 =	2·1238516	Log. 134 =	2·1271048
	1·7581226		1·7581226
	0·3657290		0·3689822
Log. sin 133°	9·8641275	Log. sin 134°	9·8569341
	10·2298565		10·2259163
L. vers. sin 133°	10·2258255	L. vers. sin. 134°	10·2290845
	— 40310		31682
			40310
		Sum	71992

71992 : 31682 :: 1° : 26′¼ nearly.

Hence 134° — 26′¼ = 133° 33′¾ = angle BOP = *bop*; therefore the versed sine *ba* = 1·689037, and BO $=\frac{AB}{1\cdot689037}$, the radius required.

Solutions were also received from Meſsrs. Harvey *and* Phelan.

XVII. QUESTION 287, *by* MECHANICUS.

By what force must a body be attracted so that, in a passing from one given point to another, the line of swiftest descent may be a circle?

SOLUTION, *by Mr.* LOWRY.

Suppose the body to have descended from A to F (fig. 65. pl. 2.) and put AE $= x$, EF $= y$, arch AF $= z$, and the force soliciting the body in the direction of the ordinate EF $= \varphi$. Then it is proved in Art. 576 of M'Laurin's Fluxions*, that "the line of swiftest descent is of such a nature, that the sine of the angle contained by it at any point F and FE, the ordinate perpendicular to the horizon, is always as the velocity of the body that descends along it at F". Now when the body descends from rest at A, the velocity at F is $= \sqrt{2\int\varphi\dot{y}}$, and the sine of the angle that the curve makes with FE, at F $= \frac{\dot{x}}{\dot{z}}$; therefore $\frac{\dot{x}}{\dot{z}} = \sqrt{\frac{2\int\varphi\dot{y}}{a}}$, or $\dot{z}^2 = \dot{x}^2\frac{a}{2\int\varphi\dot{y}}$, where a is a constant quantity.

But $\dot{z}^2 = \dot{y}^2 + \dot{x}^2$; therefore $\dot{y}^2 + \dot{x}^2 = \dot{x}^2\frac{a}{2\int\varphi\dot{y}}$, or $\dot{x}^2(a - 2\int\varphi\dot{y}) = \dot{y}^2(2\int\varphi\dot{y})$; and therefore $\dot{x} = \dot{y}\sqrt{\frac{2\int\varphi\dot{y}}{a - 2\int\varphi\dot{y}}}$, from which equation the nature of the curve may be determined when the force is given. But when the nature of the curve is given, and it is required to find the law of the attracting force, as in the question; let $\dot{x} = p\dot{y}$ be the fluxionary equation of the given curve; then $\frac{\dot{x}}{\dot{y}} = p$, and $p^2 = \frac{2\int\varphi\dot{y}}{a - 2\int\varphi\dot{y}}$, therefore $2\int\varphi\dot{y} = \frac{ap^2}{p^2+1}$, and taking fluxions, and dividing by $2\dot{y}$, we have $\varphi = \frac{ap\dot{p}}{(p^2+1)^2 \times \dot{y}}$ for the force required.

Ex. 1. When the curve is a circle, as in the question; let c be the centre, and r the radius; then, by the nature of the curve, $\dot{x} = \dot{y}\frac{y}{r-x}$; therefore $p = \frac{y}{r-x}$, $p^2 = \frac{y^2}{r^2-y^2}$,

---

[*] A neat investigation of this property of the curve of quickest descent is given at page 9 of Mr. Woodhouse's "Treatise on Isoperimetrical Problems, and the Calculus of Variations," lately published.

$p\dot{p} = \frac{r^2 y\dot{y}}{(r^2 - y^2)^2}$, $p^2 + 1 = (\frac{r^2}{r^2 - y^2})^2$;
and $\varphi = \frac{ar^2 y\dot{y}}{(r^2 - y^2)^2} \div (\frac{r^2}{r^2 - y^2})^2 \times \frac{1}{\dot{y}} = \frac{ay}{r^2}$, the force required.

Ex. 2. Let the given curve be a parabola whose parameter is $2b$; then $b\dot{x} = y\dot{y}$, and $p = \frac{y}{b}$; therefore $ap\dot{p} = \frac{ay\dot{y}}{b^2}$, $p^2 + 1 = \frac{y^2 + b^2}{b^2}$, and $\varphi = \frac{ab^2 y}{(y^2 + b^2)^2}$, the force in this case.

Ex. 3. When the given curve is a cycloid. Let r be the radius of the generating circle; then, by the nature of the curve, $\dot{x} = \dot{y} \sqrt{\frac{y}{r - y}}$; therefore $p^2 = \frac{y}{r-y}$, $p\dot{p} = \frac{r\dot{y}}{(r-y)^2}$, $p^2 + 1; = \frac{r}{r-y}$, and $\varphi = \frac{a}{r}$, a constant quantity.

When the force, instead of acting in a parallel direction, is directed to a fixed point, it appears from Art. 421, Simpson's Fluxions, that the line of swiftest descent is a circle when the attracting force is such, as to make the velocity proportional to the square of the distance of the body from the fixed point.

Solutions were received by Messrs. Higman *and* Phelan.

XVIII. QUESTION 288, *by* AMICUS.

If two tangents be drawn from the same point to a given curve, it is required to find the position of a third tangent, so that the part thereof intercepted between the tangents may be a minimum.

SOLUTION, *by Mr.* LOWRY.

Fig. 66. pl. 2. Let AB, AC be two straight lines given by position, and let it be required to draw a tangent to a given curve *m*O*n*, so that the intercepted part PQ may be a minimum.

Let *poq* be another tangent, touching the curve at *o*, a point indefinitely near to O, and meeting the other tangent in O′. With the radii O′P, O′Q and centre O′, describe the little arcs P*a*, Q*b*; then *pa* is the increment of PO′ and *qb* the decrement of QO′; therefore the increase or decrease of PQ, when removed to the position *pq*, is $= pa - qb$. But when PQ is a minimum it neither increases nor decreases, therefore $pa - qb$ must be $= 0$, or $pa = qb$.

Now because the tangents PQ, *pq* are indefinitely near to each other, the arcs P*a*, Q*b* are indefinitely small, and therefore the little triangles *pa*P, *qb*Q may be considered as rectilineal, and right angled at *a* and *b*. Hence, by trigonometry, $pa = \text{P}a \div$

$\tan \mathrm{P}p\mathrm{O}'$, and $qb = \mathrm{Q}b \div \tan \mathrm{Q}qb$; therefore $\mathrm{P}a \div \tan \mathrm{P}p\mathrm{O}' = \mathrm{Q}b \div \tan \mathrm{Q}qb$, or $\mathrm{P}a : \mathrm{Q}b\ (:: \mathrm{PO}' : \mathrm{QO}') :: \tan \mathrm{P}p\mathrm{O}' : \tan \mathrm{Q}qb$.

But since the point o is indefinitely near to O, the point O′ will also be indefinitely near to O, and therefore PO′ and QO′ will approach indefinitely near to an equality with PO and QO, and the angles $\mathrm{O}p\mathrm{P}$, $\mathrm{O}'q\mathrm{A}$ $(\mathrm{Q}qb)$ with the angles OPA, OQA. Hence by substituting PO for PO′, QO for QO′, OPA for $\mathrm{O}'p\mathrm{P}$, and OQA for $\mathrm{Q}qb$, in the above analogy, we have

$$\mathrm{PO} :: \mathrm{QO} :: \tan \mathrm{OPA} : \tan \mathrm{OQA}.$$

Therefore when the tangent PQ is a minimum, it is divided in the point of contact into segments which are proportional to the tangents of the angles that it makes with the given lines.

Innumerable other theorems, relating to the *maxima et minima*, may be discovered in a similar manner.

For example, (1). Let it be required to draw PQ so that the area APQ (or the space PRSQ, R and S being given points in AB and AC,) may be a maximum or a minimum. In this case the incremental triangle $\mathrm{O}'p\mathrm{P}$ is equal to the decremental triangle $\mathrm{O}'q\mathrm{Q}$. But these triangles are as the rectangles $\mathrm{OP}.\mathrm{O}'p$, $\mathrm{OQ}.\mathrm{O}'q$; therefore $\mathrm{OP}.\mathrm{O}'p = \mathrm{OQ}.\mathrm{O}'q$, or $\mathrm{OP}^2 = \mathrm{OQ}^2$ (because, ultimately $\mathrm{O}'p = \mathrm{OP}$ and $\mathrm{O}'q = \mathrm{OQ}$); therefore $\mathrm{OP} = \mathrm{OQ}$. Therefore when the area PAQ, or the space PRSQ, is a maximum or a minimum, the tangent PQ is bisected in the point of contact.

(2). Let it be required to draw PQ so that the sum of the lines AP, AQ may be a maximum or a minimum. In this case the increment $\mathrm{P}p$ is equal to the decrement $\mathrm{Q}q$: but $\mathrm{P}p = \mathrm{P}a \div \sin \mathrm{O}'p\mathrm{P}$, and $\mathrm{Q}q = \mathrm{Q}b \div \sin \mathrm{Q}qb$, therefore $\mathrm{P}a \div \sin \mathrm{O}'p\mathrm{P} = \mathrm{Q}b \div \sin \mathrm{Q}qb$;

or, $\mathrm{P}a : \mathrm{Q}b\ (:: \mathrm{O}'\mathrm{P} : \mathrm{O}'\mathrm{Q}) :: \sin \mathrm{O}'p\mathrm{P} : \sin \mathrm{Q}qb$,

that is, ultmately, $\mathrm{OP} : \mathrm{OQ} :: \sin \mathrm{OPA} : \sin \mathrm{OQA}$.

Therefore when the sum of the sides is a maximum or a minimum, the tangent PQ is divided in the point of contact into segments which are proportional to the sines of the angles which it makes with the given lines.

(3). Let it be required to draw PQ so that the perimeter of the triangle APQ (or $\mathrm{RP}+\mathrm{PQ}+\mathrm{SQ}$) may be a maximum or a minimum. In this case the sum of the increments $\mathrm{P}p$, pa is equal to the sum of the decrements $\mathrm{Q}q$, qb. But $pa = \mathrm{P}a \div \tan \mathrm{P}p\mathrm{O}' = \mathrm{P}a \,.\, \cot \mathrm{P}p\mathrm{O}'$, and $\mathrm{P}p = \mathrm{P}a \div \sin \mathrm{P}p\mathrm{O}' = \mathrm{P}a \,.\, \mathrm{cosec}\, \mathrm{P}p\mathrm{O}'$, therefore $\mathrm{Pp} + pa = \mathrm{P}a\ (\cot + \mathrm{cosec})\ \mathrm{P}p\mathrm{O}'$: Also $qb = \mathrm{Q}b \div \tan \mathrm{Q}qb = \mathrm{Q}b \cot \mathrm{Q}qb$, and $\mathrm{Q}q = \mathrm{Q}b \div \sin \mathrm{Q}qb = \mathrm{Q}b\ \mathrm{cosec}\ \mathrm{Q}qb$; therefore $\mathrm{Q}q + qb = \mathrm{Q}b\ (\cot + \mathrm{cosec})\ \mathrm{Q}qb$: therefore

$\mathrm{P}a\ (\cot + \mathrm{cosec})\ \mathrm{P}p\mathrm{O}' = \mathrm{Q}b\ (\cot + \mathrm{cosec})\ \mathrm{Q}qb$, and

$\mathrm{P}a : \mathrm{Q}b\ (:: \mathrm{O}'\mathrm{P} : \mathrm{O}'\mathrm{Q}) :: (\cot + \mathrm{cosec})\ \mathrm{Q}qb : (\cot + \mathrm{cosec})\ \mathrm{P}p\mathrm{O}'$,

that is, ultimately, $\mathrm{OP} : \mathrm{OQ} :: (\cot + \mathrm{cosec})\mathrm{OQS} : (\cot + \mathrm{cosec})\mathrm{OPR}$, It is obvious that the perimeter of the triangle APQ is a maximum when the curve is convex, and a minimum when it is concave towards A. Also $RP + PQ + SQ$ is a maximum when the curve is convex to RS, and a minimum when it is concave to RS.

From what has been proved above we are led to the demonstration of the following theorem, which is one of those left undemonstrated by Dr. Waring in his *Proprietates Algebraicarum Curvarum*.

Let a polygon PQRST, &c. (fig. 67, pl. 2,) of any number of sides, be described about the oval figure ABCDE, and let the sides be divided in the points of contact, so that

$PB : QB :: (\cot + \text{cosec})\, CQB : (\cot + \text{cosec})\, APB$
$CQ : CR :: (\cot + \text{cosec})\, DRC : (\cot + \text{cosec})\, BQC$
$DR : DS :: (\cot + \text{cosec})\, ESD : (\cot + \text{cosec})\, CRD$
$ES : ET :: (\cot + \text{cosec})\, ATE : (\cot + \text{cosec})\, ESD$
&c. &c.

Then the perimeter of the polygon PQRST, &c. is less than the perimeter of any other polygon of the same number of sides that can be described about the curve, and whose sides are divided in any other manner.

For suppose, when the perimeter is a minimum, that one of the sides, as PQ, is divided in some other manner in the point of contact B, and let pq be drawn so that

$bp : bq :: (\cot + \text{cosec})\, bqC : (\cot + \text{cosec})\, Apb.$

Then, by what has been proved above, $AP + pq + qC$ is less than $AP + PQ + QC$; therefore the perimeter of the polygon TpqRST, &c. is less than the perimeter TPQRST, &c. (the least of all by hypothesis), which is absurd. Therefore the perimeter cannot be a minimum when PQ is divided in any other manner than that specified in the proposition. It may be proved in the same way for any other side.

Cor. By compounding the above proportions, we have $PB \times CQ \times DR \times ES$ &c. $= BQ \times CR + DS \times ET$ &c. Therefore if a polygon PQRST &c. of any number of sides, be described about the oval figure ABCDE, so that $PB \times CQ \times DR \times ES$ &c. $= BQ \times CR \times DS \times ET$ &c. the perimeter of this polygon is less than the perimeter of any other polygon of the same number of sides that can be described about the curve. This is another of Dr. Waring's Theorems.

By a similar mode of investigation we are led to the discovery of several analogous propositions relating to figures described on the surface of a sphere.

(1.) Let AB, AC (fig. 68, pl. 2,) be two great circles given by position, and mOn a curve of any kind, given by position on the surface of a sphere: and let it be required to draw the great circle tangent POQ, so that the intercepted arch PQ may be the greatest or least possible.

Let poq be another great circle tangent indefinitely near to PQ, and intersecting it in O'. About the pole O' describe the small circles Pa, Qb; then pa is the little increment of PO', and qb the little decrement of $O'Q$; therefore the whole variation of PQ is $= pa - qb$, which in the case of a max. or a min. is $=$ to 0, or $pa = qb$. Now the little elementary triangles paP, Qbq may be considered as rectilineal, and right angled at a and b; hence $pa = Pa \div \tan O'pP$, and $qb = Qb \div \tan Qqb$; therefore $Pa \div \tan O'pP = Qb \div \tan Qqb$, or

$$Pa : Qb :: \tan O'pP : \tan Qqb.$$

But in the spherical sectors $PO'a$, $QO'b$, we have

$$Pa = \frac{\sin O'P \times \angle PO'p}{rad.}, \text{ and } Qb = \frac{\sin O'Q \times \angle QO'b^{*}}{rad.};$$

therefore $Pa : Qb :: \sin O'P : \sin O'Q$,
and $\sin O'P : \sin O'Q :: \tan O'pP : \tan Qqb$.
Therefore, ultimately (or when the points o, O', O coalesce,)

$$\sin OP : \sin OQ :: \tan OPA : \tan OQA.$$

(2). Let it be required to draw PQ so that the sum of the sides AP, AQ may be a max. or a min. In this case the increment Pp is equal to the decrement Qq. But $pP = Pa \div \sin O'pP$ and $Qq = Qb \div \sin Qqb$; therefore

$$Pa \div \sin O'pP = Qb \div \sin Qqb, \text{ or}$$

$Pa : Qb :: \sin O'pP : \sin Qqb$. But it has been shewn above that $Pa : Qb :: \sin O'P : \sin O'Q$,
therefore $\sin O'P : \sin O'Q :: \sin O'pP : \sin Qqb$;
that is, ultimately, $\sin OP : \sin OQ :: \sin OPA : \sin OQA$.

(3). Let it be required to draw PQ so that the perimeter of the triangle APQ (or the sum of the arches RP, PQ and SQ) may be a maximum or a minimum. In this case $pP + pa$ is $=$ $Qq + qb$. But $pP + pa = Pa$ (cot + cosec) $O'pP$, and $Qq + qb = Qb$ (cot + cosec) Qqb; therefore
$Pa : Qb\, (\sin O'P : \sin O'Q) :: (\cot + \text{cosec})\, Op'P : (\cot + \text{cosec})\, Qqb$,
and, ultimately,
$\sin OP : \sin OQ :: (\cot + \text{cosec})\, OpR : (\cot + \text{cosec})\, OQS$.

(4). Let it be required to draw PQ so that the area APQ, or the space $RSPQ$, may be a maximum or a minimum. In this case the little triangles $pO'P$, $QO'b$ are equal, and since pO' and qO' approach indefinitely near to an equality with PO' and QO', those triangles may be considered as isosceles, and consequently when they are equal in area, the sides $O'P$, $O'Q$ will be equal, or $OP = OQ$; hence the tangent PQ is bisected in O.

---

[*] See Vince's Trigonometry, 1st edition, Art. 178.

From what has been demonstrated the truth of the following theorems is obvious.

1. Let a great circle polygon of any number of sides be described about a curve on the surface of a sphere, so that all the sides may be bisected in the points of contact. Then the area of this polygon will be less than the area of any other polygon, of the same number of sides, that can be described about the curve, and whose sides are divided in any other manner at the points of contact.

2. Let a great circle polygon PQRST &c. of any number of sides be described about a curve on the surface of a sphere, touching it in the points A, B, C, D, E, &c. and let the sides be divided at the points of contact so that

sin PB : sin BQ :: (cot + cosec) CQB : (cot + cosec) APB
sin CQ : sin CR :: (cot + cosec) DRC : (cot + cosec) BQC
sin DR : sin DS :: (cot + cosec) ESD : (cot + cosec) CRD
sin ES : sin ET :: (cot + cosec) ATE : (cot + cosec) ESD
&c. &c.

Then the perimeter of this polygon is less than the perimeter of any other polygon of the same number of sides that can be described about the curve, and whose sides are divided in any other manner at the points of contact.

In this case sin PB × sin CQ × sin DR × sin ES &c. = sin BQ × sin CR × sin DS × sin ET &c.

The problem, in the question, may also be resolved as follows: Draw OE (fig. 66, pl. 2,) parallel to AC, and put AE $= x$, EO $= y$, EP $= v = -y\frac{\dot{x}}{\dot{y}}$ and $c =$ the cosine of PAQ; then, when PQ is a minimum, we have

$$v : x+v :: \text{PO} = \sqrt{(v^2+y^2-2cvy)} : \text{PQ} = \frac{x+v}{v}\sqrt{(v^2+y^2-2cvy)}.$$

Take the fluxion of this expression, supposing v to be constant, then

$$\frac{\dot{x}}{v}\sqrt{(v^2+y^2-2cvy)} + \frac{x+v}{v}\cdot\frac{(y-cv)\dot{y}}{\sqrt{(v^2+y^2-2cvy)}} = 0, \text{ or } \frac{(x+v)(y-cv)\dot{y}}{v^2+y^2-2cvy} = y\frac{\dot{x}}{\dot{y}} = v.$$

Therefore, by reduction, $v^2(v-cy) = xy(y-cv)$,

or, $v : x :: (y-cv)y : (v-cy)v :: \frac{y}{v-cy} : \frac{v}{y-cv}$.

But $\frac{y}{v-cy}$ is the tangent of OPA, and $\frac{v}{y-cv}$ the tangent of POE or OQA; therefore OP : OQ (:: $v : x$) :: tan OPA : tan OQA, as before.

Again, when the area is a maximum or a minimum,

$v : y :: v+x : \text{AQ} = \frac{y(v+x)}{v}$, therefore AP × PQ $= (x+v)\frac{y(v+x)}{v} =$

a max. or a min. Take the fluxion, supposing v constant, and we have

$\frac{2y}{v}(x+v)\dot{x}+(x+v)\frac{\dot{y}}{v}=o$, or $\frac{x+v}{2}=-y\frac{\dot{x}}{\dot{y}}=v$; therefore $x+v=2v$, or $x=v$, as before.

Again, when the sum of AP, PQ is amaximum or a minimum, we have $AP+PQ=v+x+\frac{y(v+x)}{v}$. Take the fluxion, supposing v constant, then $\dot{x}+\dot{y}\left(\frac{v+x}{v}\right)+y\frac{\dot{x}}{v}=o$, or $\frac{v+x}{v+y}=-\frac{\dot{x}}{\dot{y}}$; therefore $\frac{(v+x)y}{v+y}=-y\frac{\dot{x}}{\dot{y}}=v$, or $v^2=xy$; hence $v:x::y:v::$ sin OPA : sin POE (= OQA).

The problem may be resolved in the same manner when any function whatever of the lines AP, AR, PQ is to be a maximum or a minimum.

A solution to this question was received from Zeno.

XIX. QUESTION 289, *by* S. L'H.

Draw a straight line through a given point to meet a given curve PZQ, in the points P and Q, so that either the line PQ, the arch PZQ, or the area PZQP may be a minimum.

SOLUTION, *by Mr.* LOWRY.

Fig. 69, pl. 2. Let O be the given point, PQ the required line, and let PB and QC be tangents to the curve at P and Q. Draw pq indefinitely near to PQ, and with the radii OP, OQ and centre O, describe the arcs Pa, Qb. Then, it appears from the solution to the preceding question, that, when PQ is a minimum, the increment pa is equal to the decrement qb. And since pq is indefinitely near to PQ, the little triangles paP, qQb may be considered as rectilineal and right angled at a and b: therefore since pa = Pa ÷ tan OpP and qb = Qb ÷ tan Qqb, we have

Pa : Qb (:: OP : OQ):: tan OpP : tan Qqb.

But the angles OpP, Qqb (OqA) are ultimately equal to the angles OPB, OQC; therefore

OP : OQ :: tan OPB : tan OQC.

Again, when the arch PZQ is a minimum, the increment Pp is equal to the decrement Qq. But pP = Pa ÷ sin OpP and Qq = Qb ÷ sin Qqb, therefore

Pa : Qb (:: OP : OQ):: sin apP : sin Qqb,

that is, ultimately, OP : OQ :: sin OPB : sin OQC.

Again, when the area PZQP is a minimum, the incremental triangle OPp is equal to the decremental triangle OQq; and it is shewn in the very same manner as in the preceding solution that this will be the case when OP is $=$ to OQ, therefore when the area is a minimum PQ is bisected in O.

From what has been proved in this and the last solution, the truth of the following general theorems is obvious.

Let PZQ and mOn (fig. 70, pl. 2,) be two curves given by position, and let the straight line PQ touch the curve mOn at O, and meet the other given curve in P and Q, and let the angles which it makes with the curve (or the tangents) at these points, be denoted by P and Q: Then

1st. The line PQ is a minimum when
$$OP : OQ :: \tan P : \tan Q.$$

2d. The arch PZQ is a maximum or a minimum when
$$OP : OQ :: \sin P : \sin Q.$$

3d. The area PZQP is a maximum or a minimum when
OP is equal to OQ.

4th. The perimeter PZQP is a maximum or a minimum when
$$OP : OQ :: (\cot + \operatorname{cosec})\, Q : (\cot + \operatorname{cosec})\, P.$$

These propositions are also true when the figures are described on the surface of a sphere, the sines of the arcs being used instead of the lines PO and QO.

This solution, as far as it relates to the problem in the question, is not materially different from one given by S. L'Huilier, in his Principiorum Calculi Differentialis et Integralis Expositio Elementaris, p.p. 269, 270. Ex. 6, 7, 8. His solution is as follows:

1st. When the area PZQ is a minimum. (Fig. 71, pl. 2.)

Let XX', YY' be two right lines drawn so that the two corresponding areas XZX', YZY' may be equal. Then because $XZX' = YZY'$, if the common segment YZX' be taken away there will remain $XOY = X'OY'$. With the centre O and radii OY, OX' describe the arcs Yx, $X'y'$. Then because $XOY = X'OY'$,

$$\begin{aligned}
\text{or } XOY &: X'OY' = 1 : 1\\
\text{lim. } XOY &: \text{lim. } X'OY' = 1 : 1\\
\text{but lim. } XOY &: \text{lim. } YOx = 1 : 1\\
\text{lim. } YOx &: X'Oy' = OP^2 : OQ^2\\
\text{lim. } X'Oy' &: X'OY' = 1 : 1\\
\text{therefore lim. } XOY &: X'OY' = OP^2 : OQ^2.
\end{aligned}$$

Therefore when the area is a minimum, OP is $=$ OQ.

2d. When the line PQ is a minimum. (Fig. 71, pl. 2.)

Let PT and QT' be tangents to the curve at P and Q, and let $YY' = XX'$. Then because $XX' = YY'$, Xx is $= Y'y'$

$$\text{and lim. } Xx : Y'y' = 1 : 1$$
$$\text{also lim. } Xx : Yx = 1 : \tan OPT$$

$$\text{lim. } Yx : X'y' = OP : OQ$$
$$\text{lim. } X'y' : Y'y' = \tan OQT' : 1$$

Therefore $\text{lim. } Xx : Yy' = OP \tan OQT : OQ \tan OPT$.
Therefore when PQ is a minimum,

$$OP : OQ = \tan OPT : \tan OQT'.$$

3d. When the arch PZQ is a minimum. (Fig. 71, pl. 2.)
Let the arcs XZX', YZY' be equal, then $XY = X'Y'$,

$$\text{and lim. } XY : X'Y' = 1 : 1$$
$$\text{also lim. } XY : Yx = 1 : \sin OPT$$
$$\text{lim. } Yx : X'y' = OP : OQ$$
$$\text{lim. } X'y' : X'Y' = \sin OQT' : 1.$$

Therefore $\text{lim. } XY : X'Y' = OP \sin OQT' : OQ \sin OPT$.
Therefore when the arch PZQ is a minimum,

$$OP : OQ :: \sin OPT : \sin OQT'.$$

Zeno *also sent a solution to this question.*

XX. PRIZE QUESTION 290, *by Mr.* WALLACE, *R. M. College.*

If the intersection of two straight lines which contain a given angle be upon a straight line given by position, and one of them pass through a given point; what is the nature of the curve to which the other line is a normal.

FIRST SOLUTION, *by Mr.* WALLACE, *the Proposer.*

This question is only a particular case of the following general problem.

To find a curve of such a nature that the part of the axis intercepted between a given point and the normal is equal to any function of the angle which it makes with the normal.

I shall give two different solutions of this problem.

SOLUTION FIRST. Let PCQ (fig. 72, pl. 2,) be the curve whose nature is sought, AB its axis, in which A is the given point. Let CE, C'E' be any two positions of the normal, indefinitely near to one another; draw Ee perpendicular to CE. Put the angle $E = \varphi$, the line $AE = \Phi$, this symbol being put to denote some function of φ, and the normal $EC = v$. Then $EE' = d\Phi$, and $E'e = dv$. In the right angled triangle $EE'e$, $EE' : E'e :: 1 : \cos E'$, that is, $d\Phi : dv :: \cos \varphi$, therefore, $dv = \cos \varphi d\Phi$; and $v = \int \cos \varphi d\Phi$. This equation expresses the relation between the normal and the angle it makes with the axis. But we may deduce from it an equation which shall express the relation between $AD = x$, and $DC = y$, rectangular co-ordinates of the curve. For since $y = v \sin \varphi$, and $\Phi - x = v \cos \varphi$, therefore,

$$x = \Phi - \cos \varphi \times \int \cos \varphi d\Phi; \qquad y = \sin \varphi \times \int \cos \varphi d\Phi;$$

by exterminating, when it is possible, φ from these two equations, an equation will be found, expressing the relation between x and y.

To apply the general solution to the particular question under consideration, let us suppose that in fig. 73, pl. 2, AB is the straight line given by position, H the given point, and HEC the given angle, EC being the normal to the curve PQ. Draw HA perpendicular to AB, put HA $= b$, the given angle E $= \alpha$, and assuming A as the given point, let AD $= x$, DC $= y$, and EC $= v$, as in the general solution; then the angle HEA $= \alpha - \varphi$, and AE $= \frac{b\cos(\alpha-\varphi)}{\sin(\alpha-\varphi)}$; theref. $\Phi = \frac{b\cos(\alpha-\varphi)}{\sin(\alpha-\varphi)}$, and $d\Phi = \frac{b\,d\varphi}{\sin^2(\alpha-\varphi)}$, and $\cos\varphi\, d\Phi = \frac{b\cos\varphi\, d\varphi}{\sin^2(\alpha-\varphi)}$. Now $\cos\varphi = \cos\alpha\cos(\alpha-\varphi) + \sin\alpha\sin(\alpha-\varphi)$, therefore,

$$\frac{b\cos\varphi\, d\varphi}{\sin^2(\alpha-\varphi)} = \frac{b\cos\alpha\cos(\alpha-\varphi)\,d\varphi}{\sin^2(\alpha-\varphi)} + \frac{b\sin\alpha\, d\varphi}{\sin(\alpha-\varphi)},$$

and, integrating, we get

$$v = \int \cos\varphi\, d\Phi = \frac{b\cos\alpha}{\sin(\alpha-\varphi)} - b\sin\alpha \times \log.\tan\tfrac{1}{2}(\alpha-\varphi) + C,$$

this equation determines the normal v, corresponding to any point E in the axis, and if we substitute in it the values of v, $\sin(\alpha-\varphi)$, and $\tan\frac{1}{2}(\alpha-\varphi)$, which may be deduced from the two equations

$$y = v\sin\varphi, \qquad \frac{b\cos(\alpha-\varphi)}{\sin(\alpha-\varphi)} - x = v\cos\varphi,$$

we shall have an equation expressing the relation between the co-ordinates x and y.

The expression for the normal may be considerably simplified by referring the curve to a different axis: For draw Hε perpendicular to CE, and through ε draw A$\varepsilon\beta$. Then it is evident that the point H, A, ε, E are in the circumference of a circle; hence the angle BAβ is equal to EHε, which is the complement of a given angle; therefore the angle BAβ is given; and the line Aβ is given by position, and may be taken as an axis to the curve.

Now HA : Eε :: sin HEA : cos HEε, that is b : Eε :: $\sin(\alpha-\varphi)$: $\cos\alpha$, therefore E$\varepsilon = \frac{b\cos\alpha}{\sin(\alpha-\varphi)}$: Let us now put v' for the normal Cε referred to the axis Aβ, and φ' for the angle which the normal makes with this axis, then, observing that $\alpha - \varphi = 90^\circ - \varphi'$, and consequently that $\tan\frac{1}{2}(\alpha-\varphi) = \cot(45^\circ + \frac{1}{2}\varphi')$, and $\log\tan\frac{1}{2}(\alpha-\varphi) = -\log\tan(45^\circ + \frac{1}{2}\varphi')$ we get

$$v' = b\sin\alpha \times \log\tan(45^\circ + \tfrac{1}{2}\varphi') + C,$$

an equation which is more simple than the former. Since it appears that the line Hε which is perpendicular to EC always intersect it in $A\beta$, a line given by position, which we have assumed as a new axis to the curve, it follows that whatever be the magnitude of the given angle α, we need only consider the case in which it is a right angle, every other one being reducible to this.

SECOND SOLUTION We may also resolve the general problem by the theory of *Evolutes* and *Involutes*, for we can always determine by the direct method of fluxions, a curve to which the normal shall be a tangent, as is shewn in the solution of question 279. If therefore in any particular case a curve which touches all the normals be found, then, any curve which is the involute of that curve, that is, which is generated by the extremity of a thread evolved from it, will manifestly be a curve of the nature required.

If we suppose that in the question we have just now resolved, the given angle is a right angle, to which case every other is reducible, then, the curve which touches the normal in every position is a parabola as has been shewn in the first example of the second solution of question 279. Therefore the curve whose nature is sought is in every case the involute of a parabola.

SECOND SOLUTION, *by Mr.* LOWRY.

Let P (fig. 74, pl. 2,) be the given point, BA the straight line given by position, PGD the given angle, and FD the curve to which DG is a normal. Draw PC perpendicular to DG, and PB to AB, and join BC; then because of the right angles PBG, PCG, the points P, B, C, G are in the circumference of the same circle, and therefore the angle PBH is equal to the given angle I GC. Therefore the straight line HC, which passes through a given point B, and makes a given angle with a straight line PB given by position, is given by position; and if PH be drawn perpendicular to HC, H will be a given point. Let this point be taken for the origin of the abscissa HE, the line HC being considered as the axis of the curve; and put $x = \mathrm{HE}$, $y = \mathrm{ED}$, $a = \mathrm{HC}$, and $b = \mathrm{PH}$; then the subnormal EC is $= \frac{y\dot{y}}{\dot{x}}$, but EC is also $= a - x$, therefore $a - x = \frac{y\dot{y}}{\dot{x}}$. Again, because the angle PCH is the complement of the angle ECD, the right-angled triangles ECD, PCH are similar; therefore $b : a :: a - x : y$, or $by = a(a - x)$; and by completing the square, we have $a = \frac{x}{2} + \sqrt{\left(\frac{x^2}{4} - by\right)}$; this value being substituted for a, in the equation $a - x = \frac{y\dot{y}}{\dot{x}}$, we have, after proper reduction, $b\dot{x}^2 - x\dot{y}\dot{x} - y\dot{y}^2 = 0$, for the fluxionary equation of the curve.

To integrate this equation, let it be put under the form $b - x\frac{\dot{y}}{\dot{x}} - y\frac{\dot{y}^2}{\dot{x}^2} = 0$, and let $\frac{\dot{y}}{\dot{x}} = p$; then by substitution, $b - xp - yp^2 = 0$, or $y = \frac{b}{p^2} - \frac{x}{p}$: taking the fluxion, and putting for $\dot{y}$ its equal $p\dot{x}$, we have

$$p\dot{x} = -\frac{2b\dot{p}}{p^4} + \frac{x\dot{p}}{p^2} - \frac{p\dot{x}}{p^2}, \text{ or } p^4\dot{x} = (px - 2b)\,\dot{p} - p^2\dot{x};$$

therefore $\dot{x} = \frac{p}{p^4+p^2}x\dot{p} - \frac{2b\dot{p}}{p^4+p^2}$, or if $P = \frac{p}{p^4+p^2}$, and $Q = \frac{2b}{p^4+p^2}$; then $\dot{x} = Px\dot{p} - Q\dot{p}$, which equation (being of the same form) may be integrated in the same manner as that on page 40 of this volume, or page 84, vol. 2. Then we shall have x in terms of p, or p in terms of x, and then the relation of x and y may be determined by integrating the equation $\dot{y} = \dot{x}f(x)$. But since these integrations are likely to lead to complicated expressions, let us try if a simpler result can be obtained by seeking the relation between the normal CD, and the angle that it makes with the axis CH. For this purpose let CD be denoted by z, and the angle HCD by φ; then $y = z \sin\varphi$, $EC = z\cos\varphi$, $AC = b\tan\varphi$, $x = b\tan\varphi - z\cos\varphi$, $\dot{y} = \dot{z}\sin\varphi + z\cos\varphi\dot{\varphi}$, and $\dot{x} = \frac{b\dot{\varphi}}{\cos^2\varphi} - \dot{z}\cos\varphi + z\sin\varphi\dot{\varphi}$; therefore $EC = z\cos\varphi = \frac{y\dot{y}}{\dot{x}} = z\sin\varphi\left(\frac{\dot{z}\sin\varphi + z\cos\varphi\dot{\varphi}}{b\dot{\varphi} \div \cos^2\varphi - \dot{z}\cos\varphi + z\sin\varphi\dot{\varphi}}\right)$, or $\dot{z}\sin^2\varphi + z\cos\varphi\sin\varphi\dot{\varphi} = \frac{b\dot{\varphi}}{\cos\varphi} - \dot{z}\cos^2\varphi + z\cos\varphi\sin\varphi\dot{\varphi}$; therefore $\dot{z}(\sin^2\varphi + \cos^2\varphi) = \frac{b\dot{\varphi}}{\cos\varphi}$, or (because $\sin^2\varphi + \cos^2\varphi = 1$), $\dot{z} = \frac{b\dot{\varphi}}{\cos\varphi}$; and taking the fluent, we have $z = b \times \log. \tan(45^\circ + \frac{1}{2}\varphi) + c$, for the equation of the curve sought.

As another problem of this kind, let it be proposed to find the nature of the curve, that shall cut at right angles all the tangents that can be drawn to a given curve.

Let IB and BA (fig. 75, pl. 2,) be the co-ordinates of the given curve IA, and GE, ED those of the curve sought, and put $GI = m$, $IB = a$, $BA = b$, $GE = x$, $ED = y$, and the subtangent $CB = c$ (a function of a and b determinable from the equation of the curve); then by similar triangles

$b : c :: y : \mathrm{EC} = \frac{cy}{b}$, but the subnormal EC is $= \frac{y\dot{y}}{\dot{x}}$, therefore

$$\frac{cy}{b} = \frac{y\dot{y}}{\dot{x}}, \text{ or } \frac{c}{b} = \frac{\dot{y}}{\dot{x}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{also } x = m + a - \frac{cy}{b} - c \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and eliminating b and c from equation (1), by means of equation (2) and the given equation of the curve, we shall have the fluxionary equation of the curve sought.

Let, for example, the curve GIA be the common parabola, having its vertex at G, and parameter $= 2p$; then $a = \mathrm{GB}$, $b = \mathrm{BA}$, and $m = 0$. And by the property of the curve, $a^2 = 4pb$, and $\mathrm{CB} = c = \frac{1}{2} a$;

therefore $\frac{c}{b} = \frac{2p}{a}$, and $x = a - \frac{2py}{a} - \frac{1}{2} a = \frac{1}{2} a - \frac{2py}{a}$;

hence, by completing the square,

$a = x + \sqrt{(x^2 + 4py)}$ and $\frac{c}{b} = \frac{2p}{a} = \frac{2p}{x + \sqrt{(x^2 + 4py)}} = \frac{\dot{y}}{\dot{x}}$;

therefore $2p\dot{x} = \dot{y}\,(x + \sqrt{(x^2 + 4py)})$; and by reduction $p\dot{x}^2 - x\dot{y}\dot{x} - y\dot{y}^2 = 0$, the same equation as in the last problem. The curve DF is evidently the involute of the curve GIA; therefore the curve required by the question is the involute of the common parabola, H being the vertex, and P the focus (fig. 74, pl. 2.)

Again, let the given curve be the semicubical parabola of which the equation is $a^3 = \frac{27}{16} pb^2$; then the subtangent BC is $= \frac{2}{3} a$, and therefore $\frac{c}{b} = \frac{1}{2} \sqrt{\frac{3p}{a}}$; hence $x = m + a - \frac{y}{2} \sqrt{\frac{3p}{a}}$

$- \frac{2}{3} a = m + \frac{1}{3} a - \frac{y}{2} \sqrt{\frac{3p}{a}}$; or if $m = \frac{1}{2} p$,

then $x = \frac{1}{2} p + \frac{1}{3} a - \sqrt{\frac{3p}{a}}$, from whence we get $a = 3x$;

therefore $\frac{c}{b} = \frac{1}{2} \sqrt{\frac{3p}{a}} = \frac{1}{2} \sqrt{\frac{p}{x}} = \frac{\dot{y}}{\dot{x}}$, and $\frac{\dot{x}}{\sqrt{x}} + \frac{\sqrt{p}}{2} = \dot{y}$; and taking the fluent, we have $\sqrt{p} \times \sqrt{x} = y$, or $px = y^2$, an equation to the common parabola.

Innumerable problems of the same kind may be resolved in a similar manner, for let $f\,(x, y, a) = 0$, be an equation to a straight line (or a curve), a being an arbitrary quantity, and let it be required to find the nature of the curve to which this line is a normal. Then because the distance EC, or the subnormal, may be always expressed in terms of x, y and a, we shall have a function of $(x, y, a) = \frac{y\dot{y}}{\dot{x}}$, and if the arbitrary quantity a be eliminated (when it is possible) from this equation, by means of the equation $f(x, y, a) = 0$, we shall have the fluxionary equation of the curve sought.

Mr. W. WALLACE *is requested to send* to *Mr.* GLENDINNING'S *for the Medal for solving the Prize Question.*

Having a little spare room we shall insert a solution to the 245th question, which we have received from one of our correspondents, and which is rather different from any of those in the last number.

The question was "to find the fourth root of $-a$."

SOLUTION, *by* Z.

Let x be the root required. Then $x^4 = -a$, or $x^4 + a = 0$; and adding $2x^2 \surd a$ to both sides of the equation so as to make the first side a complete square, we have $x^4 + 2x^2 \surd a + a = 2x^2 \surd a$, and extracting the root, $x^2 + \surd a = \mp xa^{\frac{1}{4}} \surd 2$; or $x^2 \mp xa^{\frac{1}{4}} \surd 2 = - \surd a$. Again, completing the square, $x^2 \pm xa^{\frac{1}{4}} \surd 2 + \frac{1}{2} \surd a = -\frac{1}{2} \surd a$, and extracting the root $x \mp a^{\frac{1}{4}} \surd \frac{1}{2} = \pm a^{\frac{1}{4}} \times \frac{\surd(-1)}{2}$, or $x = \mp a^{\frac{1}{4}} \left(\frac{1 \pm \surd -1}{\surd 2}\right)$, the root required.

NOTICES.

I. *Mathematical Works lately published.*

A Third volume of Dr. **HUTTON**'s Course of Mathematics, for the use of the Royal Military Academy. This volume contains additions to some of the subjects treated of in the two former volumes; also such other new branches as appear best adapted to promote scientific education.

A Treatise on Isoperimetrical Problems and the Calculation of Variations. By **ROBERT WOODHOUSE**, A.M. F.R.S.

The Principles of Fluxions. By **WILLIAM DEALTRY**, M.A. Professor of Mathematics in the East India College.

The Philosophical Transactions of the Royal Society of London, for 1810. The Mathematical papers contained in this volume are, 1. Observations on Atmospherical Refraction, as it affects Astronomical Observations. By Mr. **GROOMBRIDGE**.—2. On the Annual Parallax of α Lyræ. By Dr. **BRINKLEY**.—3. Supplement to the first and second part of the paper of experiments, for investigating the cause of coloured concentric rings between object-glasses, and other appearances of a similar nature. By Dr. **HERSCHELL**.

An Account of the Trigonometrical Survey, carried on by order of the Master-General of his Majesty's ordnance, in the years 1800, 1, 2, 3, 4, 5, 6, 7, and 8. By Lieut. Col. **MUDGE**, of the Royal Artillery. F.R.S. and Capt. **THO. COLBY**, of the Royal Engineers. Vol. 3. 4to. 2l. 2s.

The Cambridge Problems, being a collection of the printed questions proposed to the candidates for the degree of Bachelor of Arts at the general examinations, from the year 1801 to the year 1810 inclusive. By a Graduate of the University. 6s.

Six Lectures on the Elements of Plane Trigonometry: with the method of computing Tables of Natural and Logarithmic Sines, Cosines, Tangents, &c. By the Rev. B. **BRIDGE**, Professor of Mathematics in the East India College.

An Introduction to the Theory and Practice of Mechanics, for the use of schools. By W. **MARRAT**. 8vo.

The System of Mathematical Education at the Royal Naval College, Portsmouth. Part I. By **JAMES INMAN**, A. M. Professor, Royal Naval College. 8vo. 4s.

The Elements of Linear Perspective, designed for the use of Students in the University. By D. **CRESSWELL**, A.M. Fellow of Trinity College, Cambridge.

New Principles of Linear Perspective, or the art of designing on a plane the representation of all sorts of objects, in a more general and simple method than has been hitherto done. By

Dr. **BROOK TAYLOR**, L.L.D. and R.S.S. A new edition with a portrait and life of the Author. 8vo. 14s.

Lectures on the Elements of Algebra: designed for the Use of Students of the East India College. Second edition. By Rev. B. **BRIDGE**, Professor of Mathematics in the East India College. 8vo. 7s.

A compendious and practical Treatise on the Construction, Properties, and Analogies of the three Conic Sections. By the Rev. B. **BRIDGE**. 8vo. 5s.

Elements of Geometrical Analysis, and Plane Trigonometry, with an appendix, notes, and illustrations. By **JOHN LESLIE**, Professor of Mathematics in the University of Edinburgh. 8vo. Second edition.

A Treatise on the Equilibrium of Arches, in which the theory is demonstrated upon familiar and mathematical principles. Also the method of finding the drift and shoot of the arch. By **JOS. GWILT**, Architect. 6s.

II. *Mathematical Works in the Press.*

An Elementary Investigation of the Powers and Properties of Numbers, with their Application to the Theory of Arithmetic, the Indeterminate and Diophantine Analysis, the Division of the Circle, &c. By Mr. **BARLOW**, of the Royal Military Academy Woolwich.

A General History of Mathematics, from the earliest ages to the close of the 18th century, in three octavo volumes. By **MICHAEL FRYER**, Secretary to the Bristol Philosophical Society.

An Elementary Treatise on Astronomy. By **ROBERT WOODHOUSE**, A.M. F.R.S.

III. *Foreign Books lately imported.*

Traité Elémentaire des Machines, par **HACHETTE**, 4to. plates, 2l. 12s. 6d.

Mémoire sur la Projection des Cartes Géographiques, par M. **HENRY**, 4to. 1l. 1s.

Tables Barométriques Portatives, par **BIOT**, 8vo. 4s.

Statiques des Voutes, contenant l'Essai d'une nouvelle Théorie de la poussée, par **BERARD**, 4to. plates, 15s.

Connoissance des Tems ou des Mouvemens Célestes, for 1809, 1810, 1811, and 1812, at 14s. eaeh.

Journal de l'Ecole Polytechnique, 4to. No. 10, 14s.; ditto, No. 13, 14s.; ditto, No. 14, 1l. 1s.; ditto, No. 15, 14s.

Historie de l'Astronomie depuis 1781, jusqu'a 1811, pour servir de suite à l'Historie de l'Astronomie de Bailly, par M. **VOIRON**, 4to. 1l. 10s.

Tables Astronomiques publiées par le Bareau des Longitudes de France, 4to. 18s.

Thése de Mecanique, soutenue le 9 Mai, 1811, devant la Faculté des Sciences de Paris, suivie du Programme de la Thése d'Astronomie, par **PIERRE-MARIE BOURDON**, 4to. 5s.

Introduction à la Philosophie des Mathematiques, et Technie de l'Algorithmie, par M. **HOËNÉ** de Wronski, 4to. 1l. 8s.

Mathématiques Pures, Cours Complet, par **FRANCŒUR**, 2 vol. 8vo. 1l. 10s.

IV. JOURNAL DE L'ECOLE POLYTECHNIQUE.

A Brief Account of the Memoirs and other Articles relating to Mathematics contained in the JOURNAL DE L'ECOLE POLYTECHNIQUE, *published by the Council of Instruction of that Establishment.*

The Polytechnic School, at first denominated the Central School of public works, was instituted by the French National Convention, in the second year of the Republic (1794) for the instruction of youth in the mathematical and physical sciences, previous to their being admitted into the schools of application of public service. The constitution of the school has been repeatedly changed; it however still exists, and ever since its establishment has been under the direction of the most distinguished cultivators of science in France. The work entitled "The Journal of the School" has been published at different intervals in *cahiers* or numbers, and the 15th has lately come to England. They contain the decrees of the French government relative to the constitution of the school, the plans of the different courses of instruction, and memoirs by the professors, and in some instances by the former eléves of the school on various branches of pure and mixed Mathematics, Fortification, the arts of Design, Chemistry, the Belles Letters, &c. As an enumeration of those relating to mathematics cannot but be interesting to the readers of the Repository, we proceed to give their titles in their order.

CAHIER I, commences with an article entitled *Stercotomie*, by **MONGE**, giving an account of this branch of instruction, the manner of teaching it, and the operations performed by the *eléves*. This is followed by similar articles on Civil Architecture, by **LAMBLARDIE** and **BALTARD**; on Fortification, by **DOBENHEIM**; and on Design, by **NEVEU**. Next follows a Course of Analysis applied to Mechanics, by **PRONY**, divided into lessons. Lesson 1st gives general notions of indeterminate analysis. Lesson 2d treats of equations containing two variable quantities. Lesson 3d treats of equations containing three variables, and gives an application to curve surfaces. Lesson 4th treats of first differences of functions, and gives an introduction to the method of tangents, with applications. Lesson 5th treats of first partial differences. Lesson 6th contains the notation of differences of all orders. After some articles relating to general physics, &c. the *Cahier* concludes with a memoir on the determination of shades in designs. This subject is discussed under two heads or questions:—*Quest.* 1. To find the law of the variation of the apparent intensity of light upon any given surface. *Quest.* 2. In imitating the form of a body, by painting, and knowing the law of the variation of the apparent intensity of light at its surface, to observe strictly this law in the picture. The problem which is the object of this memoir was resolved jointly by several *elèves* of the school.

CAHIER II, contains:—1. The continuation of PRONY's Lessons, viz. No. 7. Of the general relations of differences, independent of any particular relations of the variables. No. 8. Formulæ for calculating the differences of algebraic functions of a single variable. No. 9. Of the differences of transcendental functions of one variable. No. 10. gives demonstrations of the formulæ for the differences of logarithmic and exponential functions. No. 11. Demonstrations of the formulæ for trigonometrical lines in terms of the arcs. No. 12. Continuation of the subject of lesson 9th. No. 13. Formulæ of the differences of the functions of several variables, and an exposition of some properties of series deduced from the preceding theory.—2. An experimental and analytical essay on the laws of the dilatability of elastic fluids, and on the expansive force of the vapours of water and alcohol at different temperatures, by PRONY.—3. Two articles, viz. *Steréotomie*, in continuation of that in the first *Cahier*; and analysis applied to geometry, both by MONGE.—4. Notice of an elementary course of analysis, by LA GRANGE.

CAHIER III, contains:—1. Continuation of PRONY's Lesson's of Analysis. viz. No. 14. Considerations on the principles of the inverse method of differences. No. 15. Distinction between integrals and the sums of functions. Integration and summation of the powers of one variable. Of the determination and use of arbitrary constant quantities. No. 16. Integration and summation of functions composed of factors in arithmetical progression, with an application to series of figurate numbers. No. 17. Application of the method of indeterminate coefficients to the integration of rational functions. No. 18. Summation of products, one by one, two by two, &c. of a series of quantities increasing by equal intervals. No. 19. Intergals and sums of different orders. No. 20. Reciprocal transformations of imaginary transcendentals into real transcendentals. The general term of double factors which produce imaginary quantities. Determination of the factors of $x^n \pm r^n$. Cotesian theorem. Values of the powers of a cosine or sine in terms of the sine and cosine of multiple arcs. Developement of $x^n - r^n$ into a logarithmic series. Trigonometrical researches, by LAMBERT. No. 21. Introduction to the integratiou of linear equations of all orders. Application of NEWTON's differential method to the determination of the constant quantities with the condition of satisfying a certain number of observations or given values. Method of interpolation. Addition to No. 13. Differences of any order. Of the product of an exponential quantity by a variable. No. 22. The complete integration of linear equations, when the second member is equal to 0. General property of recurring series.

CAHIER IV. Continuation of PRONY's Lessons, viz. No. 23. On recurring series considered as resulting from the developement of rational fractions. Agreement of the theory which results from this consideration, with that deduced from the integral calculus. General properties of recurring series. No. 24. On the decomposition of rational fractions into partial fractions. No. 25. Treats of the complete integration of a certain equation, and of the case of integrability of another, from which is deduced the complete integration of linear equations, resolved in No. 22, when the second member instead of being $= 0$, is any function whatever of x the variable quantity. No. 26. Of differential equations in which no difference is supposed constant. On the change of a constant difference into another equal to any constant quantity. The integration of a certain assigned equation. No. 27. Considerations relating to high differentials, and the multiplicity of integrals which may be deduced from them. No. 28. Of the arbitrary functions which ought to complete integrals, &c. No. 29. Of equations to partial differences, &c. No. 30. Integration by partial differences of a certain equation. No. 31, Of a particular algorithm, founded on the analogy between the indices of differences and the exponents of powers. Fundamental theorems in the the method of differences. Transformation of formulæ for differences into those for integrals of the same order. Relations between integrals and sums, &c. No. 32. On the transition from the method of differences to that of the differential calculus, &c. A method of constructing graphically the quantities $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, &c. (when y is a function of x), and the differentials of different orders of the area of a curve. Method of interpolating, by addition and subtraction only, any number of terms between the terms of a given numerical series. On the elimination of arbitrary functions by differencing. A general theory of the osculation of curve lines. On the characters which distinguish *maxima* and *minima* in the case of any number of variables. Application of the method of tangents to the determination of the roots of equations. This lesson terminates the direct and inverse method of differences.

CAHIER V, contains:—1. An account of PRONY's Course of Mechanics, given in the 5th year of the Republic. 2. Memoir on statics, containing the demonstration of the principle of virtual velocities, and the theory of moments, by FOURIER. 3. An essay of numerical analysis on the transformation of fractions, by LA GRANGE*. 4. On the principle of virtual velocities, by LA GRANGE. 5. On the determination of a plane which remains always parallel to itself in the motion of a system of bodies acting in any manner whatever on one another and free from all external action, by LA PLACE, 6. Description and use of a dynamometer, an instrument for discovering and comparing the relative force of men, of horses, &c. also for estimating the resistance of machines and the moving powers to be applied to them, by REGNIER. 7. On the principle of virtual velocities and the decomposition of circular motions, by PRONY.

CAHIER VI, contains:—1. An address by PRONY to his scholars at the commencement of his course of analysis. 2. A discourse on the object of the theory of analytic functions, by LA GRANGE. 3. A discourse relating to the teaching of descriptive geometry, by GAYVERNON, 4. Solutions of some problems relative to spherical triangles, with a complete analysis of these triangles, by LA GRANGE†. 5. Theory of the motion about a free axis of rotation, of a body of an invariable figure, urged by any power, by PRONY. 6. On mechanics, by LA PLACE. 7. On curves of double curvature, taken from the works of MONGE.

CAHIERS VII and VIII, contain the first three parts of a work by PRONY, intitled "*Mécanique Philosophique ou Analyse Raisonnée des diverses parties de la Science de l'Equilibre et du Mouvement.*"

CAHIER IX, which is to contain the sequel of PRONY's work, has not come to this country, and probably is not yet published.

CAHIER X, contains:—1. Solutions of several problems in geometry, by M. BRIANCHON. 2. Memoir on polygons and polyhedrons, by **Poinsot**. The subjects treated in this memoir are referred to what is called the *geometry of situation.* 3. Note, on the application of the theory of particular solutions of differential equations to some questions relating to the art of the engineer.

---

* This essay has been given in the Mathematical Repository, Vol. I. Part III. Page 24.

† This memoir has been given in the Mathematical Repository, Vol. I. Part III. Page 1.

Cahier XI. contains:—1. Memoir upon the curve surface, all the normals of which are tangents to the same sphere, by **Monge**. 2. Memoir on the curve surface, the normals to which are tangents to the surface of a cone with an arbitrary base. 3. A detailed analysis of different questions relating to the motion of a body urged by any powers, by **Prony**. 4. Memoir on the application of algebra to geometry, by **Monge** and **Hachette**, with a supplement by **Hachette** and **Poisson**. 5. Memoir on the plurality of integrals in the calculus of differences, by **Poisson**. 6. Considerations on the integrals of equations to finite differences, by **Biot**. 7. Memoir on elimination in algebraic equations, by **Poisson**. 8. Memoir on the theory of the motion of projectiles in a resisting medium, by **Moreau**. 9. Memoir on Gnomonicks, by **Le François**.

Cahier XII, contains twenty lessons on the calculus of functions, by **La Grange**. These, with the addition of two, have also been published in an octavo form, as a distinct work.

Cahier XIII, contains:—1. Memoir of the curve surface, all the normals of which are tangents to the same *developable* surface*, by **Monge**. 2. Memoir on the curve surface which envelopes the space run by a sphere of a variable radius, and of which the centre describes any curve of double curvature, by **Monge**. 3. Memoir on particular solutions of differential equations and of equations to differences, by **Poisson**. 4. Memoir on equations to mixed differences, by **Poisson**. 5. Researches into some points of the theory of derivative functions, which lead to a new demonstration of **Taylor**'s series, and to the finite expression for the terms which are neglected, when the series stops at any term, by **Ampere**. 6. On the composition of moments, and the composition of areas, by **Poinsot**. 7. General theory of the equilibrium and of the motion of systems, by **Poinsot**. 8. Memoir on axes of synchronous suspension, by **Biot**. 9. General demonstration of the principle of virtual velocities, independent of the consideration of infinitely small quantities, by **Ampere**. 10. Formulæ for passing from a system of rectangular co-ordinates to a system of oblique co-ordinates, and to another system of rectangular co-ordinates. Properties of surfaces of the second degree referred to their conjugate diameters. On the contact of conic surfaces of the second order, by LIVET. 11. On curve surfaces of the second degree, by BRIANCHON.

---

* A developable surface is that which may be expanded upon a plane, without tearing, or duplicature. See Essais de Géometrie, No. 96, by **La Croix**.

CAHIER XIV, contains:—1. Memoir on opticks, by MALUS. 2. Essay on the description of lines and surfaces of the second degree. On the determination of radii of curvature and tangents to lines of curvature, by DUPIN. 3. Sequel of the memoir on optics, by MALUS. 4. Extract from lessons given at the Polytechnic school on singular points. (that is, multiple points, conjugate points, &c.) of curves, by POISSON. 5. Memoir on the oscillations of pendulums in a resisting medium, taking into account the extensibility of the thread, or rod of the pendulum, by POISSON. 6. Memoir on the advantages which result in the theory of curves from the consideration of osculating parabolas, with reflections on the differential functions, the values of which do not change by the transformation of the axes, by AMPERE. 7. Memoir on the transformation of co-ordinates, by FRANÇOIS. 8. Memoir on the integration of partial differential equations, with a supplement, by BRISSON. 9. Memoir on the possibility of substituting the hydraulic ram (bélier hydraulique) for the old machine of Marly, by MONTGOLFIER. 10. Memoir on the theory of sound, by POISSON. 11. Two lessons on the calculus of functions, in addition to those in Cahier xii. by LA GRANGE.

CAHIER XV, contains:—1. Memoir on the secular inequalities of the mean motion of the planets, by POISSON. 2. Explanation of a particular difficulty which occurs in the calculation of the attraction of spheroids differing very little from a sphere, by LA GRANGE. 3. Essay of the application of analysis to some questions of elementary geometry, by MONGE. 4. Construction of the equation of vibrating cords, by MONGE. 5. On the method of the greatest common divisor applied to elimination, by BRET. 6. Memoir on the motion of rotation of the earth, by POISSON. 7. Memoir on the measure of the refracting power of opaque bodies, by MALUS. 8. Memoir on different points of analysis, by LA PLACE, viz. 1st. On the calculus of generating functions. 2d. On the definite integrals of equations to partial differences. 3d. On the reciprocal passage of real to imaginary results. 4th. On the integration of equations to finite differences, not linear. 5th. On the reduction of functions into tables. 9. Memoir on the variation of arbitrary constant quantities in questions concerning mechanics, by POISSON. 10. Addition to the memoir on pendulums, printed in the 14th Cahier, by POISSON.

ARTICLE III.

Solutions to Questions proposed in Number X.

I. QUESTION 291, *by Mr.* GERMAN BUXTON, *London.*

To find two numbers whose difference is a square, and if any number (n) be added to each, or to their sum, they will make three other squares.

FIRST SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let x and y be the numbers sought; then by the question $x - y$, $x + n$, $y + n$, and $x + y + n$ are to be squares. Assume $x + n = v^2$, and $y + n = w^2$; then $x = v^2 - n$, and $y = w^2 - n$, and, consequently,

$$x - y = v^2 - w^2,$$

$$\text{and } x + y + n = v^2 + w^2 - n..$$

We have therefore only to make $v^2 - w^2$ and $v^2 + w^2 - n$ into squares. Now $v^2 - w^2$ will evidently be a square when $v = r^2 + s^2$, and $w = r^2 - s^2$; or when v and w are equal to those quantities multiplied or divided by any number, constant or indeterminate, positive or negative, r and s being any numbers whatever. But the assumption most convenient for our purpose will be $v = \frac{r^2 + s^2}{2ms}$, and $w = \frac{r^2 - s^2}{2ms}$; and these values being substituted in the second expression, it becomes $\frac{2r^4 + 2s^4}{4m^2 s^2} - n$, which is to be a square; and, therefore, (multiplying by $4m^2 s^2$), $2r^4 + 2s^4 - 4m^2 s^2 n$ must be a square. It would be difficult, however, to resolve this expression in its present form, but if we put $r = s + t$ it will be transformed into $4s^4 + 8s^3t + (12t^2 - 4m^2n)\, s^2 + 8st^3 + 2t^4$, which is a complete biquadratic, having the co-efficient of the first term a square, and therefore may be resolved by known methods. For this purpose we must assume for its root, such a function of s, t and m, as will make all the powers of s higher than the first, disappear from the equation, and then s may be determined in terms of t, m and n. This will be accomplished by taking $2s^2 + 2st + (2t^2 - m^2n)$ for the root; then the square will be

$$4s^4 + 8s^3t + (12t^2 - 4m^2n)\, s^2 + 8st^3 - 4stm^2n + (2t^2 - m^2n)^2,$$

which being equated with the above expression, there will result $2t^4 = -4stm^2n + (2t^2 - m^2n)^2$; from whence we get

$$s = \frac{2t^4 + m^4n^2}{4tm^2n} - t;\quad r = \frac{2t^4 + m^4n^2}{4tm^2n};$$

therefore we have

$$x = \left(\frac{r^2 + s^2}{2ms}\right)^2 - n, \text{ and } y = \left(\frac{r^2 - s^2}{2ms}\right)^2 - n,$$

where $r = \frac{2t^4 + m^4n^2}{4tm^2n}$ and $s = \frac{2t^4 + m^4n^2}{4tm^2n} - t$; t and m being any numbers taken at pleasure.

If we suppose $m = t = 1$, then $r = \frac{2 + n^2}{4n}$ and $s = \frac{2 + n^2}{4n} - 1$ and if $n = 3$, then $x = 22\frac{121}{144}$, $y = 22$, $x - y = (\frac{11}{12})^2$, $x +$ [illegible] $= (\frac{61}{12})^2$, $y + n = 5^2$, $x + y + n = (\frac{83}{12})^2$.

If $n = 4$, $x = 22\frac{17}{64}$, $y = 21$, $x - y = (\frac{9}{8})^2$, $x + n = (\frac{41}{8})^2$ $y + n = 5^2$, $x + y + n = (\frac{55}{8})^2$.

If m be $= 2$ and $t = 1$, then $r = \frac{1 + 8n^2}{8n}$ and $s = \frac{1 + 8n^2}{8n} - 1$; hence if $n = 1$, $x = 5\frac{145}{256}$, $y = 5\frac{1}{4}$, $x - y = (\frac{9}{16})^2$, $x + n = (\frac{41}{16})^2$, $y + n = (\frac{5}{2})^2$, $x + y + n = (\frac{55}{16})^2$.

If we put $v = r^2 + s^2$, and $w = r^2 - s^2$, the expression $v^2 + w^2 - n$ will be transformed into $2r^4 + 2s^4 - n$, or $4s^4 + 8s^3t + 12s^2t^2 + 8st^3 + 2t^4 - n$, putting $r = s + t$. And if $2s^2 + 2st + 2t^2$ be assumed for the root, we shall find that the terms containing s, on both sides of the equation, will destroy each other, and there will remain only $2t^4 - n = 4t^4$, so that nothing can be determined from this assumption respecting the value of s. We are led however to a curious enough conclusion, for since $2t^4 - n = 4t^4$, $-n$ is $= 2t^4$; therefore it follows that, if n be the double of any biquadrate number t^4, and x be taken $= (2r^2 - 2rt + t^2)^2 + n$, and $y = (2rt - t^2)^2 + n$; then $x - y$, $x - n$, $y - n$ and $x + y - n$ will be squares, when r and t are any numbers whatever.

If, for example, we suppose $r = 2$ and $t = 1$, or $n = 2t^4 = 2$, then $x = (2r^2 - 2r + 1)^2 + 2 = 27$, and $y = (2r - 1)^2 + 2 = 11$.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Let $a^2 - n$ and $b^2 - n$ denote the two required numbers; it is obvious that this notation answers two of the conditions, that is, n being added to either will make a square. It then only remains to find such expressions for a and b, as will make $a^2 + b^2 - n$ and $a^2 - b^2$ squares. Put $a^2 + b^2 - n = (a + b - s)^2$, which gives $a = \frac{s^2 - 2bs + n}{2(s - b)}$. Again, put $a^2 - b^2 = (rb - a)^2$, and this gives $a = \frac{b(r^2 + 1)}{2r}$; therefore $\frac{b(r^2 + 1)}{r} = \frac{s^2 - 2bs + n}{s - b}$, hence $b^2 - \frac{sb(r + 1)^2}{r^2 + 1} = \frac{-r(s^2 + n)}{r^2 + 1}$; and completing the square, &c.

$b = \frac{s(r+1)^2 \pm \sqrt{(s^2r^4 - 4nr^3 + 6s^2r^2 - 4nr + s^2)}}{2(r^2+1)}$. From whence it appears that in order to have b rational, $s^2r^4 - 4nr^3 + 6s^2r^2 - 4nr + s^2$ must be a rational square.

Put $s^2r^4 - 4nr^3 + 6s^2r^2 - 4nr + s^2 = (sr^2 - \frac{2nr}{s} - s)^2$; whence $r = \frac{2ns^2}{2s^4 - n^2}$. Writing $sr^2 - \frac{2nr}{s} - s$, in the place of the radical part in the above expression for b, we have $b = \frac{r}{s} \times \frac{s^2(r+1) - n}{r^2+1} = 2sn \times \frac{2s^6 - s^2n^2 + n^3}{4s^8 + n^4}$, or $b = \frac{s^2(r+1) + nr}{s(r^2+1)} = \frac{s(2s^4 - n^2)(2s^4 + 2ns^2 + n^2)}{4s^8 + n^4}$: using the latter expression for b, $a = \frac{b(r^2+1)}{2r} = \frac{2s^4 + 2ns^2 + n^2}{4ns}$. Therefore $\frac{(2s^4 + 2ns^2 + n^2)^2}{16n^2s^2} - n$, and $\frac{s^2(2s^4 - n^2)^2(2s^4 + 2ns^2 + n^2)^2}{(4s^8 + n^4)^2} - n$ are general expressions for the required numbers, in which n, may denote any given number, and s be taken at pleasure.

Other general expressions for the numbers, might have been deduced from the foregoing conclusions, but those put down are sufficient.

THIRD SOLUTION, *by Mr.* PETER PHELAN, *Carlow, Ireland.*

Put a and $a + x^2$ for the two numbers; then $a + n$, $a + n + x^2$, and $2a + n + x^2$ must be squares. Put $a + n = w^2$; then $a = w^2 - n$; hence $w^2 + x^2$, and $2w^2 - n + x^2$ must be squares. Assume $w + p$ for the root of the first; then $w^2 + 2wp + p^2 = w^2 + x^2$, $w = \frac{x^2 - p^2}{2p^2}$, and $2w^2 = \frac{(x^2 - p^2)^2}{2p^2}$; hence $\frac{(x^2 - p^2)^2}{2p^2} - n + x^2 =$ a square. That is $\frac{x^4}{2p^2} - \frac{2p^2x^2}{2p^2} + \frac{p^2}{2} - n + x^2$, or its equal $\frac{x^4}{2p^2} + \frac{p^2}{2} - n$ must be a square, therefore multiplying by $4p^2$, $2x^4 + 2p^4 - 4p^2n$ must be a square. Put $x = p + s$; then we have $2x^4 + 2p^4 - 4p^2n = 4p^4 + 8p^3s + (12s^2 - 4n)p^2 + 8ps^3 + 2s^4 =$ a rational square. Put $2p^2 + 2ps + (2s^2 - n)$ for the root, and equate its square to the above expression, from whence we get $(2s^2 - n)^2 - 4psn = 2s^4$; hence p comes out

$\frac{n^2 - 4s^2n + 2s^4}{4sn}$, and from this the numbers are easily found.

Messrs. Cavill, Harvey *and* Watts, *and* Kyn *answered it.*

II. QUESTION 292, *by Mr.* Cunliffe, *R. M. College.*

If a quadrantal arc of a circle, radius 1, is divided into any four parts, the sum of the products of the tangents of every two of these parts is equal to 1 + the continued product of all the four tangents. Required proof?

Solution, *by Mr.* Joseph Williams.

Let $\alpha, \beta, \gamma, \delta$, be the four parts and their respective tangents A, B, C, D; then since $\alpha + \beta + \gamma + \delta = 90°$,

$$\tan(\alpha+\beta) = \tan(90° - (\gamma - \delta)) = \cot(\gamma+\delta) = \frac{1}{\tan(\gamma+\delta)}:$$

therefore, (by the formula for the tangent of the sum of two arcs, page 22, of Mr. Woodhouse's Trigonometry.) $\frac{A+B}{1-A \cdot B} = \frac{1-C \cdot D}{C+D}$;
therefore $A \cdot C + A \cdot D + B \cdot C + B \cdot D = 1 - A \cdot B - C \cdot D + A\,B \times C \cdot D$,
and $A \cdot C + A \cdot B + A \cdot D + B \cdot C + B \cdot D + C \cdot D = 1 - A \cdot B \cdot C \cdot D$.

Solutions were also received from Messrs, Cavill, Eratosthenes, Kyn *and* Phelan.

III. QUESTION 293, *by Mr.* Dawes, *Birmingham.*

In what latitude on the longest, day will the shadow of a point of a gnomon, describe a right-angled hyperbola upon an horizontal plane.

First Solution, *by Mr.* Phelan.

Let MZO (fig. 76, pl. 3,) be the hyperbola described in the required latitude on the summer solstice; V the top of the gnomon or vertex of the cone formed by the intercepted rays. Produce the abscissa NZ to meet the side QV of the cone in Y, then will ZY be the transverse axis of the hyperbola.

Let MO be the ordinate to the abscissa NZ, and QOPM a circular section of the cone passing through M, O. Describe a circle through the points V, Y, Z, and draw VW parallel to the diameter PQ and let it meet the circumference in X. Then the triangles VYW, QYW are evidently similar, as are also the triangles VWZ, NPZ; hence

WY : WV :: NY : NQ, and WZ : WV :: NZ : NP; therefore $WY \times WZ : WV^2 :: NY \times NZ : NQ \times NP$. But by the property of the circle $NQ \times NP = MN \times NO = MN^2$; and by the property of the right-angled hyperbola $NY \times NZ = MN^2$; therefor $WY \times WZ = WV^2 = WV \times WX$; and consequently $WV = WX$. But WV bisects the angle YVZ; therefore the angle XVZ is equal to the angle XVY or to its equal XZW, and the angle at X is common to the triangles ZXW, ZXV, therefore they are similar, and $VX : XZ :: XZ : WX = \frac{1}{2} VX$; hence $VX = XZ \sqrt{2}$. Now the angle WVY = the angle WVZ is the sun's declination = 23° 28′, and the angle VWY is the co-latitude, then by trigonometry, $XZ : VX :: 1 : \sqrt{2} :: \sin 23° 28' : \sin \angle VZX = \angle VWY \doteq 34° 17'$ the complement of the latitude.

Second Solution, *by Mr.* Cunliffe.

I shall first find the equation of the curve described by the shadow of the top of a pole, upon an horizontal plane, and from thence deduce the answer to the question.

Let HP (fig. 77, pl. 3,) represent a pole perpendicular to the horizon, VN a portion of the curve described by the shadow of the top of the pole upon the horizontal plane HVN: draw the meridional line HV meeting the curve in V. Let HN represent a position of the shadow of the pole, and draw NM perpendicular to HV; also draw PN.

Put $HP = h$, $HM = x$ and $MN = y$, then $PN = \sqrt{(h^2 + x^2 + y^2)}$ and $HN = \sqrt{(x^2 + y^2)}$. By the principles of trigonometry, the sine of the sun's altitude will be expressed by $h \div \sqrt{(h^2 + x^2 + y^2)}$, and its cosine by $\sqrt{(x^2 + y^2)} \div \sqrt{(h^2 + x^2 + y^2)}$, radius 1; also the sine of the suns azimuth VHN will be expressed by $y \div \sqrt{(x^2 + y^2)}$, and its cosine by $x \div \sqrt{(x^2 + y^2)}$.

In the spherical triangle ZPS (fig. 78, pl. 3,) the side ZP represents the complement of the latitude; the side PS the complement of the declination, or sun's polar distance, and the side ZS the complement of the sun's altitude. Then the angle PZS will represent the sun's azimuth.

Put $\sin ZP = c$, $\cos PS = d$, radius 1; and let the cosine of the altitude, azimuth, &c. be expressed as above. Then by a well known theorem

$$\sin ZP \times \sin ZS \times \cos PZS + \cos ZP \times \cos ZS = \cos PS,$$

and this expression by the foregoing notation becomes

$$cx \sqrt{(x^2+y^2)} \div \sqrt{(h^2+x^2+y^2)} \sqrt{(x^2+y^2)} + h\sqrt{(1-c^2)} \div \sqrt{(h^2+x^2+y^2)} = d,$$

that is $cx + h\sqrt{(1-c^2)} = d\sqrt{(h^2 + x^2 + y^2)}$,

or $d^2 y^2 = x^2 (c^2 - d^2) + 2chx \sqrt{(1-c^2)} + h^2 (1 - c^2 - d^2)$.

This equation expresses the property of a conic section, as is very well known; therefore a conic section is the curve described by the shadow of the top of a pole upon an horizontal plane.

When c, is greater than d, the curve will be an hyperbola, when equal to d, a parabola; and when less than d, an ellipse.

When the co-efficients of y^2 and x^2 on opposite sides of the equation are equal to each other, viz. $c^2 - d^2 = d^2$, or $c = d\sqrt{2}$, the curve will be an equilateral hyperbola; these things are too well known to need proving here.

Upon the longest day, when d = sin 23° 28′ then c = sin of 34° 16′ 29″ the complement of the latitude 55° 43′ 31″.

Third Solution, *by Mr.* Lowry.

Several writers have shewn that the curve described by the shadow of an object, upon the plane of the horizon, during the sun's apparent diurnal revolution is a conic section. This has generally been done by conceiving the rays intercepted by the highest point of the object to form a conic surface, the intersection of which with the plane of the horizon forms the curve required. The axes, parameter &c. are then determined from the properties of the cone. See Simpson's Fluxions, page 545, and Mathematical Dissertations, page 155. In what follows I propose to deduce the equation of the curve from the principles of trigonometry independent of the properties of the cone, and then resolve the particular case specified in the question.

Let CH (fig. 79. pl 3.), drawn perpendicular to the horizon, represent the gnomon, and let CA be its shadow when the sun is on the meridian, CP its shadow at some other time, and PAQ the curve required. Join PH, and AH, and draw PB perpendicular to CA. Then will the angle PCA be the azimuth, and the angle CPH the altitude of the sun when the shadow of the point of the gnomon is at P. Put z = PC, v = PH, h = CH, φ = the angle PCA and θ = the angle CPH; then if we can determine the relation between φ and θ in terms of z and v, and known quantities, so as to exterminate θ and v from the equation, we shall have the relation between z and φ. For this purpose, conceive a spherical triangle to have for its angular points, the zenith, the pole and the sun's place; then the side between the zenith and the sun will be the co-altitude, or distance of the sun from the zenith; the side between the zenith and pole will be the co-latitude; the side between the sun and pole the co-declination; and the angle at the zenith the sun's azimuth. Put a = the co-altitude, b = the co-declination, and c = the co-latitude; then by the common rule for finding an angle from the sides, we have

$$\frac{\cos b - \cos c \cos a}{\sin c \sin a} = \text{the cosine of the azimuth} = \cos\varphi.$$

But $\sin a = \cos\theta = z \div v$, and $\cos a = \sin\theta = h \div v$, therefore substituting these values of sin a and cos a in the above equation, we have $\frac{v\cos b}{z\sin c} - \frac{h\cos c}{z\sin c} = \cos\varphi$, and if we exterminate v, by means of the equation $v^2 = z^2 + h^2$, we shall have the equation of the curve in terms of z and φ, It will be more convenient for our purpose, however, to have the equation in terms of the rectangular co-ordinates CB and PB or x and y, Now $\cos\varphi = x \div z$, therefore by substitution and dividing by z, we have

$$v\cos b - h\cos c = x\sin c, \text{ or } v\cos b = h\cos c + x\sin c,$$

and squaring both sides, in order to exterminate v, we get

$$(x^2+y^2+h^2)\cos^2 b = h^2\cos^2 c + 2hx\cos c\sin c + x^2\sin^2 c,$$

or $\cos^2 b y^2 + x^2(\cos^2 b - \sin^2 c) - 2hx\cos c\sin c + h^2(\cos^2 b - \cos^2 c) = 0$, which is an equation to an ellipse, a hyperbola, or parabola, according as the co-efficient of x^2 is positive, negative or nothing; and a right-angled hyperbola when the co-efficients of y^2 and x^2 are equal; therefore the curve is an ellipse when the declination is greater than the co-latitude; an hyperbola when it is less, and a parabola when it is equal to the co-latitude. In the rectangular hyperbola, $\cos^2 b = \sin^2 c - \cos^2 b$, or $\sin c = \cos b\sqrt{2}$, and therefore in the question, where $\cos b = \sin 23° 28'$, we have the logarithmic sine of $c = 9{\cdot}7506331$; hence $c = 34° 16'\frac{1}{4}$ nearly; and the latitude $= 55° 43'\frac{1}{2}$.

The axes may be determined from the equation, by finding the value of x when $y = 0$, in which case we have

$$x^2 - \frac{2h\cos c\sin c}{\cos^2 b - \sin^2 c}x = -h^2\left(\frac{\cos^2 b - \cos^2 c}{\cos^2 b - \sin^2 c}\right),$$

and by completing the square, and extracting the root,

$x = \dfrac{h\sin b\cos b}{\cos^2 b - \sin^2 c} \pm \dfrac{h\cos c\sin c}{\cos^2 b - \sin^2 c}$, the sum of which values of $x = \dfrac{2h\sin b\cos b}{\cos^2 b - \sin^2 b}$ is evidently equal to the transverse axis, and the distance of the point C from the centre of the curve $= \dfrac{h\cos c\sin c}{\cos^2 b - \sin^2 c}$; hence by substituting this value for x we have the conjugate axis $= \dfrac{2h\sin b}{\sqrt{(\cos^2 b - \sin^2 c)}}$.

IV. QUESTION 294, *by* G. V.

Find three square numbers in arithmetical progression, the sum of the square roots of which shall be a cube.

FIRST SOLUTION, *by Mr.* LOWRY.

Let x^2, y^2, and z^2 be the numbers required; then by the nature of arithmetical progression, $2y^2 - x^2 = z^2$. Put $y - v = x$, then $y^2 + 2yv - v^2 = z^2$. Assume $y^2 + 2yv - v^2 = (\frac{s}{r}v - y)^2$; then $y^2 + 2yv - v^2 = z^2 = \frac{s^2}{r^2}v^2 - \frac{2s}{r}vy + y^2$, or $2y - v = \frac{s^2}{r^2}v - \frac{2s}{r}y$, and by reduction $y = \frac{(r^2+s^2)v}{2r^2-2rs}$; therefore $x = y - v = \frac{(2rs + s^2 - r^2)v}{2r^2-2rs}$, and $z = \frac{s}{r}v - y = \frac{(2rs + r^2 - s^2)v}{2r^2 - 2rs}$. These values of x, y and z are the roots of three squares in arithmetical progression, whatever the values of r, s, and v may be; and it is evident that they will be equally so if the common denominator $2r^2 - 2rs$ be rejected; we may therefore take $x = (2rs + s^2 - r^2)v$, $y = (r^2 + s^2)v$, and $z = (2rs + r^2 - s^2)v$; and the sum of the three roots is then $= (r^2 + 4rs + s^2)v$, which is to be a cube. Now there are several ways in which this last condition may be satisfied, but the most obvious is, to make $v = (r^2 + 4rs + s^2)^2$, and then we shall have

$$x = (2rs + s^2 - r^2)\ (r^2 + 4rs + s^2)^2,$$
$$y = (r^2 + s^2)\ (r^2 + 4rs + s^2)^2,$$
$$\text{and } z = (2rs + r^2 - s^2)\ (r^2 + 4rs + s^2)^2,$$

where r and s may be taken at pleasure.

If $r = 2$ and $s = 1$, then $x = 169$, $y = 845$, $z = 1183$, and the three squares are 169^2, 845^2 and 1183^2, which I believe are the least whole numbers that will answer the question.

The same answered otherwise.

Putting as before, x^2, y^2 and z^2 for the numbers, we have $2y^2 = x^2 + z^2$, or $y^2 - x^2 = z^2 - y^2$, and by resolving the expression into factors $(y - x)\ (y + x) = (z - y)\ (z + y)$.

Assume $n(y - x) = m(z - y)$, and $m(y + x) = n(z + y)$;

then $ny - nx = mz - my$, and $y = \frac{mz + nx}{m + n}$.

Also $my + mx = nz + ny$, and $y = \frac{nz - mx}{m - n}$;

$$\text{hence } \frac{mz + nx}{m + n} = \frac{nz - mx}{m - n},$$

or, $(m - n)\,mz + (m - n)\,nx = (m + n)\,nz - (m + n)\,mx$; and by reduction $(2nm + m^2 - n^2)\,x = (2nm + n^2 - m^2)z$, which equation will be satisfied by making $x = 2nm + n^2 - m^2$, and $z = 2nm + m^2 - n^2$; and then y is $= m^2 + n^2$. These values of x, y and z, are the roots of three squares in arithmetical progression whatever numbers are taken for m and n; and their sum is $= m^2 + 4mn + n^2$, which is to be a cube. Suppose now that $m^2 + 4mn + n^2$ is the product of the two factors

$m + an$ and $m + bn$, then $m^2 + 4mn + n^2 = m^2 + (a+b)\,mn + abn^2$, and to render these expressions indentical we must have $a + b = 4$ and $ab = 1$. Now it is evident that the expression $(m + an)(m + bn)$, and consequently $m^2 + 4mn + n^2$, will be a cube when each of its factors is a cube; let us therefore assume $m + an = (r + as)^3$, and $m + bn = (r + bs)^3$, then $m + an = r^3 + 3r^2as + 3ra^2s^2 + a^3s^3$, and if for a^2 and a^3 we substitute their values $4a - 1$, and $15a - 4$, drawn from the equations $a + b = 4$, and $ab = 1$, by exterminating b, we shall have

$$m + an = (r^3 - 3rs^2 - 4s^3) + (3r^2s + 12rs^2 + 15s^3)a,$$

and in like manner by cubing $(r + bs)$, and substituting for b^2 and b^3 their values found from the above equations, by exterminating a, we get

$$m + bn = (r^3 - 3rs^2 - 4s^3) + (3r^2s + 12rs^2 + 15s^3)b,$$

and it is evident that these two equations will be satisfied by making $m = r^3 - 3rs^2 - 4s^3$, and $n = 3r^2s + 12rs^2 + 15s^3$. We have therefore

$x = 2nm + n^2 - m^2$, $y = m^2 + n^2$, and $z = 2mn + m^2 - n^2$, where $m = r^3 - 3rs^2 - 4s^3$ and $n = 3r^2s + 12rs^2 + 15s^3$; r and s being any numbers whatever.

The question may be resolved in a similar manner when the sum of the roots is to be an nth power, by putting $m + an = (r + as)^n$ and $m + bn = (r + bs)^n$.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

It is well known that $v(q^2 - p^2 + 2pq)$, $v(p^2 + q^2)$ and $v(p^2 - q^2 + 2pq)$ express the roots of three squares in arithmetical progression, and their sum is $v(p^2 + 4pq + q^2)$, which put $= a^3$, whence $v = a^3 \div (p^2 + 4pq + q^2)$. In order to have v an integer, put $a = s(p^2 + 4pq + q^2)$, then $v = a^3 \div (p^2 + 4pq + q^2) = s^3(p^2 + 4pq + q^2)$: whence $v(q^2 - p^2 + 2pq) = s^3(p^2 + 4pq + q^2)^2\,(q^2 - p^2 + 2pq)$; $v(p^2 + q^2) = s^3(p^2 + 4pq + q^2) \cdot (p^2 + q^2)$; and $v\,(p^2 - q^2 + 2pq) = s^3(p^2 + 4pq + q^2) \cdot (q^2 - p^2 + 2pq)$, which are general expressions for the roots of three squares in arithmetical progression, the sum of the roots being a cube: in which expressions, p, q and s may be taken at pleasure.

Ex. Take $p = 2$, $q = 1$ and $s = 1$; then the roots are 169, 845 and 1183.

It may be remarked, that the sum of the roots $v(p^2 + 4pq + q^2)$ may be put equal to any number whatever; therefore any number whatever is capable of being divided into three parts, whose squares will constitute an arithmetical progression, in an infinite variety of ways.

THIRD SOLUTION, *by Mr.* ABRAHAM KYN.

If x^2, y^2 and z^2 are the numbers, then $x + y + z = a^3$. By the nature of arithmetical progression $y^2 - x^2 = z^2 - y^2$; therefore $(y - x)(y + x) = (z - y)(z + y)$.

Suppose $y + x = m(z - y)$ and $m(y - x) = z + y$; then from the first equation $x = m(z - y) - y$, and from the second $x = \frac{mv - y - z}{n}$; therefore $m(z - y) - y = \frac{my - y - z}{n}$,

and $z = \left(\frac{m^2 + 2n - 1}{m^2 + 1}\right)y$; therefore $x = mz - my - y = \frac{m^2 - 2m - 1}{m^2 + 1}y$,

and $x + y + z = \frac{(3m^2 - 1)y}{m^2 + 1} = a^3$; therefore $y = \left(\frac{m^2 + 1}{3m^2 - 1}\right)a^3$.

If $m = 3$, and $a = 1$, $y = \frac{5}{13}$, $x = \frac{1}{13}$, $z = \frac{7}{13}$ whose squares $\frac{1}{169}$, $\frac{25}{169}$ and $\frac{49}{169}$ have a common difference.

FOURTH SOLUTION, *by Mr.* WALLACE, *R. M. College.*

Let x^2, y^2 and z^2, be the three squares sought. By the question $x^2 + z^2 = 2y^2$; therefore $2y^2 - x^2 = z^2$, a square.

To reduce this expression to the known theory of indeterminate equations of the second degree, let us assume $y = x' + y'$, $x = x' + 2y'$, and then we have $2y^2 - x^2 = x'^2 - 2y'^2 = z^2$. Now as every expression of the form $x^2 + cy^2$ may be made a square by assuming $x = p^2 - cq^2$, and $y = 2pq$, (EULER's Algebra, Part II. Art. 182), in the present case we must assume $x' = p^2 + 2q^2$, $y' = 2pq$, and then we have

$$x = p^2 + 2q^2 + 4pq = (p + 2q)^2 - 2q^2,$$
$$y = p^2 + 2q^2 + 2pq = (p + q)^2 + q^2,$$
$$z = p^2 - 2q^2,$$

and $x + y + z = 3p^2 + 2q^2 + 6pq = 3(p + q)^2 - q^2$.

We must now make $3(p + q)^2 - q^2$, or putting $p' = p + q$, $3p'^2 - q^2$ a cube. Now there are two known methods by which every expression of the form $ap^2 + cq^2$ may be made a cube. (See EULER's Algebra, Part II. Chap. 12.)

It will be a cube if we assume

$$p = m(am^2 + cn^2), \qquad q = n(am^2 + cn^2),$$

m and n being any quantities whatever.

It will also be a cube if we assume

$$p = m(am^2 - 3cn^2), \qquad q = n(3am^2 - cn^2),$$

and in either case $ap^2 + cq^2 = (am^2 + cn^2)^3$.

Applying these formulæ to the question under consideration, we may assume

$$p' = m(3m^2 - n^2), \qquad q = n(3m^2 - n^2) \ldots\ldots\ldots\ldots \text{(1)}$$

or we may assume

$$p' = m(3m^2 + 3n^2), \qquad q = n(9m^2 + n^2) \ldots\ldots\ldots\ldots \text{(2)}$$

and, deducing thence the values of x, y and z, we shall have a solution of the problem.

By employing the first set of formulæ we get

$$x = \left\{(m+n)^2 - 2n^2\right\}.\left\{3m^2 - n^2\right\}^2,$$
$$y = \left\{m^2 + n^2\right\}\left\{3m^2 - n^2\right\}^2,$$
$$z = \left\{(m-n)^2 - 2n^2\right\}\left\{3m^2 - n^2\right\}^2.$$

If we mean to have z a positive integer, m must not be less than $3n$.

Let us suppose $m = 3$, $n = 1$, then

$$x = 2^3 . 13^2 . 7,\ y = 2^3 . 13^2 . 5,\ z = 2^3 . 13^2,$$

but as these numbers are each multiplied by a cube, it is evident that by rejecting that factor we shall have a smaller set of numbers which will equally satisfy the question, viz.

$x = 13^2 . 7$, $y = 13^2 . 5$, $z = 13^2$ and $x^2 = 1183^2$, $y^2 = 845^2$, $z^2 = 169^2$.

We might determine other values of x, y and z from the second set of expressions for p' and q. It would seem however that they would be more complex than those given above.

Solutions were also received from Messrs. Cavill, Eratosthenes, Harvey *and* Watts, Phelan *and* Scurr.

V. QUESTION 295, *by Mr.* W. WALLACE, *R. M. College.*

THEOREM. In any conic section if normals be drawn at the extremities of a chord passing through the focus so as to intersect one another, the sum of the squares of the parts of the normals between the curve and their intersection shall have a given ratio to the square of the chord. Required the demonstration.

FIRST SOLUTION, *by Mr.* PETER PHELAN.

Let the straight line PQ (fig. 80, pl. 3), drawn through the focus M of an ellipse or hyperbola, meet the curve in P and Q, and let normals be drawn to the curve at these points to intersect in O; also draw the tangents PA, QA to intersect in A. Let S be the other focus and join SO and SA. Also draw SP and SQ; then because the normals bisect the angles SPQ and SQP, the line SO will bisect the angle QSP; also because the line drawn from the point of intersection of any two tangents to either focus, bisects the angle contained by the lines drawn from the same focus to the points of contact, SA bisects the angle QSP; therefore the points S, O and A are in the same straight line. Produce SP and SQ to V and N, till PV = MP, and QN = QM, and join VA, NA and MA; then will SN = SV = the transverse axis, and it is well known (see solution to quest. 145) that the lines AV, AM, and AN are equal to each other, and perpendicular to the lines SV, PQ and SN respectively. Produce AM to meet OW, drawn parallel to PQ, in W, and draw OX perpendicular to PQ; then, because of the right angles at P, W and Q, the points A, P, W, O and Q are in a circle; therefore because the chords PQ and WO are parallel, the angle WQP = the angle OPQ, and since OX = WM, the right-angled triangles WMQ and OXP are equal in every respect; therefore MQ = PX, and consequently PM = QX: Hence, by the property of the circle, $OX \cdot AM = WM \cdot MA = PM \cdot MQ$. But $OP^2 + OQ^2 = QX^2 + PX^2 + 2OX^2 = PQ^2 - 2QX \cdot PX + 2OX^2$; and, by a well known property of the conic sections, $QX \cdot PX = (PM \cdot MQ =)\ PQ \cdot p$, where p = a fourth part of the parameter (Emerson's Conics, B. I. P. 68, and B. II. P. 65, Cor. 3); therefore $OP^2 + OQ^2 = PQ^2 - 2PQ \cdot p + 2OX^2$. Draw OZ perpendicular to SN, then because OQ bisects the angle SQP, OZ is = OX, and therefore ZN = PQ: hence by similar triangles $SN : SZ = SN \mp PQ :: AN = AM : OZ = OX :: 2OX \cdot AM = 2(PM \cdot MQ) = 2PQ \cdot p : 2OX^2$,

$$\text{therefore } 2OX^2 = \frac{SN \mp PQ}{SN} \times 2PQ \cdot p = 2PQ \cdot p \mp \frac{2PQ^2 \cdot p}{SN}$$

$$\text{and } OP^2 + OQ^2 = PQ^2 \pm \frac{2PQ^2 \cdot p}{SN},$$

therefore $OP^2 + OQ^2 : PQ^2 :: SN \pm 2p : SN$, a given ratio. The upper sign being for the ellipse and the lower one for the hyperbola. When the curve is a parabola, the angle PAQ formed by the tangents is a right one, and consequently the angle QOP formed by the normals is also a right one; therefore the sum of the squares of OP and OQ is in that case equal to the square of PQ.

SECOND SOLUTION, *by Mr.* WALLACE, *the Proposer.*

Let CA (fig. 81. pl. 3.) be the semi-transverse axis, F the focus, B*b* a chord passing through the focus, BD, *bd* normals intersecting each other in H, and meeting the axis in D and *d*: Draw HE parallel to the axis meeting the chord in E.

By a known property of the conic sections, CF : CA (:: DF : FB) :: HE : EB and CF : CA (:: *d*F : F*b*) :: HE : E*b* (Emerson's Conics, B. I. Prop. 11 Cor. 3. and B. II. Prop. 10. Cor. 4); therefore EB = E*b*, and $2BE^2 : 2EH^2 :: CA^2 : CF^2$, and consequently $2BE^2 + 2EH^2 : 4BE^2 :: CA^2 + CF^2 : 2CA^2$, that is (Elements of Geometry) $BH^2 + bH^2 : Bb^2 :: CA^2 + CF^2 : 2CA^2$, therefore $BH^2 + bH^2$ has to Bb^2 a given ratio.

VI. QUESTION 296, *by Mr.* CUNLIFFE.

Suppose V*m*R a curve in a vertical plane, and P a given point in its axis; to find its nature so that the pressure of a given weight *m*, attached to a string *m* P, fastened at P, and resting on the curve, may be every where the same.

FIRST SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Draw the vertical line PV (fig. 82. pl. 3.) for the axis of the curve, V being the vertex or highest point thereof. In PV produced take PO to OV in the given ratio of the weight *m* to its pressure against the curve. With the centre O, and radius OV, describe a circle which will be the required curve. For, suppose *m* to represent any position of the weight suspended by a string P*m*, and resting against the curve; and draw the radius O*m*. By a known property of the circle, the radius O*m* is perpendicular to the curve at *m*. Therefore by the composition of forces, the weight *m*, its pressure against the curve at *m*, and the tension of the string P*m*, will be respectively as the lines PO, O*m* and P*m*, and in the several directions of those lines. That is the weight *m*, is to its pressure against the curve at *m*, in the constant ratio of PO to O*m*, and consequently the pressure of the weight *m* against the periphery of the circle, will be every where the same, whatever is the length of the suspending string P*m*, within the proper limits.

Remark. It is manifest that the length of the suspending string P*m*, must be greater than PV and less than PV + 2OV.

SECOND SOLUTION, *by Mr.* LOWRY.

Let *m*O (fig. 82. pl. 3.) be a normal to the required curve at *m*, and let the vertical line *mg* represent the given weight or force of gravity; draw *gp* parallel to *m*P, then *pm* will represent the pressure of the weight against the curve, and which is, by the question, to be the same for every position of the string; and consequently must have a constant ratio to the given weight. Therefore *pm* must have to *gm* a constant ratio, or because of the similar triangles *pmg* and O*m*P, O*m* must have to OP a constant ratio. We have therefore to find a curve V*m*R such, that the normal O*m* shall have a given ratio to the segment OP intercepted between the normal and a given point in the axis. This is the same as question 288, vol. ii, and it might be a sufficient answer to the question to refer to the solutions there given, and where it is shewn, that either a straight line or a circle will answer the proposed conditions; but since the fluxional equation which occurs in the solution may be integrated differently and rather more elegantly than is there done, I shall again go through the whole of the solution.

Let *m*E be drawn perpendicular to PO, and put PE $= x$, E*m* $= y$, and the ratio of *m*O : OP $= 1 : n$; then the normal

MO is $= y\sqrt{(1+\frac{\dot{y}^2}{\dot{x}^2})}$, and the subnormal XO $= \frac{y\dot{y}}{\dot{x}}$; therefore, by the question,

$$y\sqrt{(1+\frac{\dot{y}^2}{\dot{x}^2})} : x + \frac{y\dot{y}}{\dot{x}} :: 1 : n,$$

$$\text{or } x + \frac{y\dot{y}}{\dot{x}} = ny\sqrt{(1+\frac{\dot{y}^2}{\dot{x}^2})}.$$

To integrate this equation put $\frac{\dot{y}}{\dot{x}} = p$, then it becomes

$$x + py = ny\sqrt{(1+p^2)},$$

and, taking the fluxions, we have

$\dot{x} + p\dot{y} + y\dot{p} = n\dot{y}\sqrt{(1+p^2)} + \frac{nyp\dot{p}}{\sqrt{(1+p^2)}} =$ (by reducing the terms on the right hand side of the equation to a common denominator), to $\left(\frac{\dot{y}}{p} + p\dot{y} + y\dot{p}\right)\frac{np}{\sqrt{(1+p^2)}}$. But $\dot{y} = p\dot{x}$, or $\dot{x} = \frac{\dot{y}}{p}$, therefore

$$\dot{x} + p\dot{y} + y\dot{p} = (\dot{x} + p\dot{y} + y\dot{p})\frac{np}{\sqrt{(1+p^4)}}$$

$$\text{or } (x+py)^{\cdot} \times \left(1 - \frac{np}{\sqrt{(1+p^2)}}\right) = 0; \text{ which gives}$$

$(x+py)^{\cdot} = 0$, and $1 - \frac{np}{\sqrt{(1+p^2)}} = 0$. From the first of these equations, $x + py = a$, or $p = \frac{a-x}{y}$, (a being an arbitrary quantity), which being substituted for p in the original equation, we have $a = ny\sqrt{(1+\left(\frac{a-x}{y}\right)^2)}$, and by reduction, $y^2 + (a-x)^2 = n^2a^2$, which is an equation to a circle whose radius is na.

From the second equation $1 - \frac{np}{\sqrt{(1+p^2)}} = 0$, we get $p = \frac{1}{\sqrt{(n^2-1)}}$, which being substituted in the original equation we have $x + \frac{y}{\sqrt{(n^2-1)}} = ny\sqrt{(1+\frac{1}{n^2-1})}$, or $x = y\sqrt{(n^2-1)}$, an equation to a straight line.

Hence it appears that if a straight line P*m*S be drawn to make an angle with the vertical line PO, so that its tangent may be $= \sqrt{(n^2 - 1)}$, and any circle VmR (whatever) be described to touch the line PmS, and have its centre in the vertical line: Then either the straight line or circle will answer the conditions of the question; for the pressure of the weight, at any point, in either of them, will evidently be the same whatever the length of the string may be.

Messrs. Eratosthenes, Harvey *and* Watts *also answered it.*

VII. QUESTION 297, *by* G. V.

Required the nature of a curve such that a tangent being drawn from any point in the curve to meet the axis, the distance of the point of contact from a given point in the axis shall have a given ratio to the segment of the axis between that point and the tangent.

FIRST SOLUTION, *by Mr.* LOWRY.

Let P (fig. 83. pl. 3.) be the given point in the axis, and CA the curve required; then by the question, if CT be a tangent to any point of the curve, CP has to TP a given ratio, suppose that of 1 to n.

Put PD $= x$ and DC $= y$, then the subtangent TD $= y\frac{\dot{x}}{\dot{y}}$, and TP $= y\frac{\dot{x}}{\dot{y}} - x = \frac{y\dot{x} - x\dot{y}}{\dot{y}}$; therefore, since $n \times$ CP $=$ TP, $n\sqrt{(x^2 + y^2)} = \frac{y\dot{x} - x\dot{y}}{\dot{y}}$, or $ny\dot{y}\sqrt{\left(\frac{x^2}{y^2} + 1\right)} = y\dot{x} - x\dot{y}$.

Let both sides of this equation be divided by y^2, then we shall have $\frac{n\dot{y}}{y}\sqrt{\left(\frac{x^2}{y^2} + 1\right)} = \frac{y\dot{x} - x\dot{y}}{y^2} = \left(\frac{x}{y}\right)^{\cdot}$, therefore $n\frac{\dot{y}}{y} = \left(\frac{x}{y}\right)^{\cdot} \times \frac{1}{\sqrt{\left(\frac{x^2}{y^2} + 1\right)}}$; and taking the fluents, $n(\log. y - \log. c)$ $= \log. \left(\frac{x}{y} + \sqrt{\left(\frac{x^2}{y^2} + 1\right)}\right)$, c being the value of y when $x = 0$;

and, passing from logarithms to numbers, $\frac{y^n}{c^n} = \frac{x}{y} + \sqrt{\left(\frac{x^2}{y^2} + 1\right)}$;

hence $\frac{y^{n+1}}{c^n} - x = \sqrt{(x^2 + y^2)}$, and $\left(\frac{y^{n+1}}{c^n}\right)^2 - 2x\left(\frac{y^{n+1}}{c^n}\right) = y^2$;

therefore $x = \frac{y^{n+1}}{2c^n} - \frac{c^n}{2y^{n-1}}$, the equation of the curve.

If $n = 1$, then $x = \frac{y^2}{2c} - \frac{c}{2}$, or $y^2 = c(2x + c)$, an equation to the common parabola, of which P is the focus and c the semi-parameter.

SECOND SOLUTION, *by Messrs.* KYN *and* WILLIAMS.

Let AC (fig. 83. pl. 3.) be the required curve, P the given point in the axis, and C any point in the curve. Let AP $= a$, PD $= x$, CD $= y$; then, by the question PT : PC :: n : 1, or PT $= n \times$ PC. But TD $= \frac{dxy}{dy}$, from the expression for the subtangent.

$$\therefore \text{PT} = \text{DT} - \text{DP} = \frac{dxy}{dy} - x = \frac{dxy - xdy}{dy}.$$

And PC $= \sqrt{(x^2 + y^2)}$ $\therefore$ $\frac{dxy - xdy}{dy} = n\sqrt{(x^2 + y^2)}$.

Let $x = zy$ $\therefore$ $dx = dzy + zdy$,

$\therefore$ by substitution $dzy^2 + zydy - zydy = ndy\sqrt{(z^2y^2 + y^2)}$

$$\therefore dzy^2 = nydy\sqrt{(z^2 + 1)} \quad \therefore \frac{dz}{\sqrt{(z^2 + 1)}} = \frac{ndy}{y},$$

$\therefore$ h.l. $(z + \sqrt{(1 + z^2)}) = n$ h.l. y = h.l. $y^n + c^n$.

Let $y = b$ when $x = 0$, and $\therefore$ $z = 0$,

$$\therefore \text{h.l.}(z + \sqrt{(1 + z^2)}) = \text{h.l.}\frac{y^n}{b^n} \quad \therefore z + \sqrt{(1 + z^2)} = \frac{y^n}{b^n},$$

$$\therefore \frac{x}{y} + \frac{\sqrt{(x^2 + y^2)}}{y} = \frac{y^n}{b^n} \quad \therefore \sqrt{(x^2 + y^2)} = \frac{y^{n+1}}{b^n} - x$$

$$\therefore x^2 + y^2 = \frac{y^{2n+2}}{b^{2n}} - \frac{2xy^{n+1}}{b^n} + x^2$$

$$\therefore \frac{2xy^{n+1}}{b^n} = \frac{y^{2n+2}}{b^{2n}} - y^2 \quad \therefore x = \frac{y^{n+1}}{2b^n} - \frac{b^n}{2y^{n-1}}.$$

If $n = 1$, $x = \frac{y^2}{2b} - \frac{b}{2}$ $\therefore$ $y^2 = 2bx + b^2 = (2x + b)b$ an equation to a parabola. Also in this case PC $=$ PT, a property of the parabola, where P is the focus.

Eratosthenes *and Mr.* Phelan *also answered it.*

VIII. QUESTION 298, *by* Mr. LOWRY, *R. M. College.*

To determine a curve that shall cut an indefinite number of curves of the same given species, having a common axis, so that all the tangents drawn to the given curves at the points of intersection may tend to the same given point.

First Solution, by *Mr.* Wallace, *R. M. College.*

Let A (fig. 84. pl. 3.) be the given point, and PQ any one of the curves given by position to which AB is a tangent at B. Through A draw any straight line AC given by position: and draw BD perpendicular to AC; and supposing the curves given in kind, as well as that which is required, to be referred to AC as an axis, let AD $= x$ and DB $= y$ be their common co-ordinates.

Let us suppose that the nature of the curves given in kind is defined by the equation $\text{V} = 0$, where V is put for an expression of calculation, involving, besides the co-ordinates x and y, and constant quantities, a parameter p which is constant for any one of the curves, but which has different values in different curves. The conditions of the problem shew that $dx : dy :: x : y$, but from the equation $\text{V} = 0$ we have $\left(\frac{d\text{V}}{dx}\right) dx + \left(\frac{d\text{V}}{dy}\right) dy = 0$, therefore $dx : dy :: -\left(\frac{d\text{V}}{dy}\right) : \left(\frac{d\text{V}}{dx}\right)$, and hence $x : y :: -\left(\frac{d\text{V}}{dy}\right) : \left(\frac{d\text{V}}{dx}\right)$: Thus it appears that the nature of the curve required is expressed by the two equations

$$\text{V} = 0; \quad \left(\frac{d\text{V}}{dx}\right) x + \left(\frac{d\text{V}}{dy}\right) y = 0;$$

each of these contain the indeterminate quantity p, but by elimination we may obtain a single equation involving only the co-ordinates x, y and given quantities, which will be the equation of the curve whose nature is required.

Ex. Let us suppose the curves given in kind to be parabolas, each having its axis PF parallel to a line given by position, which line we may assume to be AC. From P, the vertex of any one of the parabolas, draw PE perpendicular to AC. Let BD meet PF in G. Put AE $= a$, EB $= b$, the parameter $= p$. Then by the nature of the figure $p \times$ PG $=$ BG^2, that is, in symbols, $p(x - a) = (y - b)^2$; hence we have

$$\text{V} = p(x - a) - (y - b)^2$$

$$\left(\frac{d\text{V}}{dx}\right) = p, \quad \left(\frac{d\text{V}}{dy}\right) = 2(y - b).$$

Therefore the nature of the curve which is the locus of B will be expressed by these two equations

$$p(x - a) - (y - b)^2 = 0 \ldots\ldots\ldots\ldots(1)$$

$$px - 2y(y - b) = 0 \ldots\ldots\ldots\ldots(2)$$

by which we may resolve three distinct problems.

First, we may suppose that the parabolas, besides having their axes parallel to AC, have also a common vertex given by position, then, to determine the nature of the curve we have only to eliminate p (which is indeterminate) from these equations, and we get for the equation of the curve

$$(2a - x)(y + b) = 2ab,$$

which shews it to be a hyperbola. The asymptotes are manifestly parallel to the co-ordinates, and the curve passes through the given point A as well as through P the common vertex of the parabolas: And if X and Y denote the co-ordinates of the centre of the hyperbola, it is easy to see that $X = +2a$ and $Y = -b$.

Secondly, we may suppose that the parabolas have all the same parameters, but that their vertices are at different points in a line parallel to AC: or we may suppose that the axis of a given parabola slides along the line PF given by position and parallel to AC. Then in this case p is constant and a an indefinite quantity. Now the equation (2) is already free from this quantity and gives us

$$\tfrac{1}{4}px = y\,(y - b)$$

for the equation of the curve, from which it is easy to see that it is a parabola.

Lastly, we may suppose the vertex of a given parabola to be carried in a line perpendicular to the line AC. In this case b is indeterminate and must be exterminated from the equations (1) and (2), which done we get

$$px^2 = 4\,(x - a)y^2$$

an equation to a line of the third order.

SECOND SOLUTION, *by Mr.* LOWRY, *the Proposer.*

Suppose the given curves to be defined by the general equation $f(x, y, a) = 0$, where x and y are co-ordinates, and a an arbitrary quantity, which is supposed to be constant for all the points in the same curve, but to vary from curve to curve so as to form a series of curves differing insensibly from one another. Then, by the question, it is required to find a curve that shall cut all these curves in such a manner that a tangent drawn to any one of them at the point of intersection shall pass through a given point. Let AB be the axis of the given curve AC (fig. 85. pl. 3.) P the given point, and CC′ the curve required: Draw CD and PR perpendicular and PE parallel to AB. Put $b =$ PR $=$ ED, $c =$ RA, $x =$ AD and $y =$ DC; then $\dot{x} : \dot{y}$ :: PE : CE :: $c + x : y - b$, and therefore $\dfrac{\dot{x}}{\dot{y}} = \dfrac{c + x}{y - b}$.

Also by taking the fluxion of the expression $f(x, y, a)$ considering a as constant, and exterminating a from the equation which results, by means of the equation $f(x, y, a) = 0$, we shall obtain $\frac{\dot{x}}{\dot{y}}$ = a function of x and $y = f(x, y)$, and shall then have $\frac{c+x}{y-b} = f(x, y)$ for the equation of the curve in terms of x and y, independent of a as it ought to be.

Example 1. Let the given curves be a series of parabolas having their vertices at A, and formed from the equation $ax - y^2 = 0$ (which corresponds with the general equation $f(x, y, a) = 0$), by taking successively all possible values for a; then, taking the fluxion, we have $a\dot{x} = 2y\dot{y}$, or $\frac{\dot{x}}{\dot{y}} = \frac{2y}{a}$; but $a = \frac{y^2}{x}$, therefore $\frac{\dot{x}}{\dot{y}} = \frac{2x}{y} = f(x, y)$, and we have $\frac{c+x}{y-b} = \frac{2x}{y}$; or, by reduction, $xy - 2bx - cy = 0$, which is an equation to the common hyperbola.

To construct the curve, make AI = AB = c, and on IK, drawn perpendicular to AB, take IF = $2b$, and draw FH parallel to AB; then will FK and FH be the asymptotes to the hyperbola, and $\sqrt{2bc}$ the semi-transverse axis. For if FH meet DC in S, then FS = $x - c$, and CS = $y - 2b$; therefore FC × CS = $(x-c)(y-2b) = xy - 2bx - cy + 2bc$: but $xy - 2bx - cy = 0$, and therefore FC × CS = $2bc$, as it ought to be.

When $b = 0$, or the given point is in the axis of the parabola, the equation becomes $x - c = 0$, and the straight line IK will then cut the curves in the manner required.

Example 2. Let the curves be a series of ellipses described on the axis AB when the other axis varies. Put d = the constant axis AB, and a = the variable one; then, by the property of the curve, $a^2(dx - x^2) - d^2y^2 = 0$, and taking the fluxion, $\frac{\dot{x}}{\dot{y}} = \frac{2d^2y}{a^2(d-2x)}$; therefore, since $a^2 = \frac{d^2y^2}{dx - x^2}$, we have, by substitution, $\frac{\dot{x}}{\dot{y}} = \frac{2dx - 2x^2}{y(d-2x)}$ and consequently $\frac{c+x}{y-b} = \frac{2dx - 2x^2}{y(d-2x)}$, or by reduction,

$dcy + 2bdx - (2c+d)xy + 2bx^2 = 0$, which is also an equation to an hyperbola.

When $b = 0$, or the given point is in the axis of the curve, the equation becomes

$$dcy - (2c+d)xy = 0, \text{ or } dc - (2c+d)x = 0,$$

and $x = \frac{dc}{2c+d}$; therefore if a point I be found in the axis AB such, that c may be to $c+d$, as AI to IB, and a line IK be drawn perpendicular to AB, it will cut the ellipses in the manner required.

EXAMPLE 3. Instead of one of the axes being constant we may suppose both of them to vary; but then they must have such a relation that the one may be expressed in terms of the other. For example, let them have the given ratio of 1 to n; that is, let $d = an$;

then $\frac{\dot{x}}{\dot{y}} = \frac{2n^2y}{d-2x}$, and since $a^2 = \frac{d^2y^2}{dx-x^2}$, or $1 = \frac{n^2y^2}{dx-x^2}$,

we have $d = \frac{n^2y^2+x^2}{x}$, $d-2x = \frac{n^2y^2-x^2}{x}$, and $\frac{\dot{x}}{\dot{y}} = \frac{2n^2yx}{n^2y^2-x^2}$;

therefore $\frac{c+x}{y-b} = \frac{2n^2yx}{n^2y^2-x^2}$, and by reduction

$cn^2y^2 - cx^2 + n^2xy^2 - x^3 + 2bn^2yx = 0$, an equation to a line of the third order. When $n = 1$, or the curve is a circle, the equation becomes $cy^2 - cx^2 - xy^2 - x^3 + 2byx = 0$, and when $b = 0$, or the given point is in the axis, we have $c(y^2 - x^2) - xy^2 - x^3 = 0$. For the construction of the curve, in this case, see Maclaurin's Algebra, pages 312 and 313.

The question may be resolved in a similar manner when any property whatever of the tangent, subtangent, normal, line making a given angle with the curve, &c. is known, that will enable us to find the value of $\frac{\dot{x}}{\dot{y}}$ in terms of x, y, and known quantities.

Suppose, for example, that the tangent CR (fig. 86. pl. 3.) instead of passing through a given point, were required to be of a given length; then

$\dot{y} : \sqrt{(\dot{x}^2+\dot{y}^2)} :: y : \text{CR} = \frac{y}{\dot{y}}\sqrt{(\dot{x}^2+\dot{y}^2)}$ = a constant quantity c,

therefore $y^2(\dot{x}^2 + \dot{y}^2) = c^2\dot{y}^2$, and $\frac{\dot{x}}{\dot{y}} = \frac{\sqrt{(c^2-y^2)}}{y}$;

consequently we have $\frac{\sqrt{(c^2-y^2)}}{y} = f(x, y,)$, for the equation of the curve. In the parabola, where $\frac{\dot{x}}{\dot{y}} = \frac{2x}{y}$, we have $\frac{\sqrt{(c^2-y^2)}}{y} = \frac{2x}{y}$, or $4x^2 + y^2 - c^2 = 0$, an equation to an ellipse, of which the axes are $\frac{1}{2}c$ and $2c$, and centre is A.

As another example, let it be required to find the nature of the curve, when the tangent is to touch a given circle.

Let P be the centre of the given circle and PH (fig. 87. pl. 3.)

its radius: through P draw KPE meeting the ordinate CD in E, and the tangent CH in K, and draw PR perpendicular to the axis AB. Put HP $= r$, PR $= b$, AR $= c$, PE $= v$, EC $= u$, and KP $= z$; then KH $= \sqrt{(z^2 - r^2)}$, and by similar triangles KH : HP :: KE : CE $= \frac{r(z+v)}{\sqrt{(z^2 - r^2)}} = u$; therefore $r^2(z+v)^2 = u^2(z^2 - r^2)$, and by completing the square, we have $z = \frac{r}{u^2 - r^2}(rv + u\sqrt{(u^2 + v^2 - r^2)})$, therefore KE $= z + v = \frac{vu^2 + ru\sqrt{(u^2 + v^2 - r^2)}}{u^2 - r^2}$, and since $\frac{\dot{x}}{\dot{y}} = \frac{z+v}{u}$, we have $\frac{\dot{x}}{\dot{y}} = \frac{vu + r\sqrt{(u^2 + v^2 - r^2)}}{u^2 - r^2} = f(x, y)$; or putting for v and u their values $x + a$ and $y - b$ we have

$$\frac{(x+a)y - (x+a)b + r\sqrt{(y^2 + x^2 - 2by - 2ax + b^2 + a^2 - r^2)}}{y^2 - 2by + b^2 - r^2} = f(x,y)$$

for the general equation of the curve. The question may be resolved in a similar manner when the tangent is required to touch any other given curve, or when it is required to cut two straight lines given by position so, that the rectangle of the segments between the points of intersection and given points in the lines may be equal to a given space.

Messrs. Harvey *and* Watts *sent an answer to this question.*

IX. QUESTION 299, *by* HERMODORUS.

Given the vanishing line its centre and distance of an original plane and the projection of a straight line, and of a point in that plane; through that point to draw a straight line so that the angle made by the two straight lines shall be equal to that made by their originals.

SOLUTION, *by* HERMODORUS, *the Proposer.*

Fig. 88. pl. 3. Let the given vanishing line be HD, X its centre, XA perpendicular to HD equal to the given distance, FD the given indefinite projection of a straight line, and G the given point.

Let FD meet HD in D and join AD, in AX produced take XY $=$ XA and join YD, suppose GF to be the line required making with FD the angle GFD which is to be equal to that it represents: produce GF to meet HD in H and join AH, then (by the principles of perspective) the angle HAD is equal to that represented by GFD. Join HY then the angle HYD $=$ the angle HAD; therefore the angle HYD $=$ the angle HFD, whence

Y, F, H, D are in a circle, therefore the angle GHY is equal to the angle FDY.

CONSTRUCTION. Find Y as in the analysis and join GY upon which describe the segment of a circle containing an angle equal to that contained by FD and YD: if it meets the line HD in two points either of them solves the problem, if it touches HD the point of contact is the only solution, if it neither touches nor cuts HD the problem is impossible. If FD coincides with YD and G with Y the problem is indeterminate.

X. QUESTION 300, *by* AMICUS.

To find curves that have the following property, viz. that if P be a given point in the straight line APB given by position and any straight line whatever be drawn through P to meet the curve in two points, and tangents be drawn at these points to meet the line AB in E and F, the segments PE, PF shall always be equal.

SOLUTION, *by* A. B.

Let the straight line drawn through P, (fig. 89, pl. 3,) meet the curve in M and N, and draw the ordinates MR, NS to the axis AB. Put PR $= x$, and MR $= y$, then the subtangent ER $= -\frac{ydx}{dy}$, and PE $= -$ PF $= -\frac{ydx}{dy} + x$ (which must be a function of $\frac{x}{y}) = f\left(\frac{x}{y}\right)$. But $-\frac{ydx}{dy} + x = \frac{-\frac{ydx - xdy}{y^2}}{-\frac{dy}{y^2}}$ $= \frac{-d\frac{x}{y}}{d\frac{1}{y}}$, therefore $\frac{-d\frac{x}{y}}{d\frac{1}{y}} = f\left(\frac{x}{y}\right)$; whence $\frac{1}{y} = \int \frac{-d\frac{x}{y}}{f\left(\frac{x}{y}\right)} =$ $\mathrm{F}\left(\frac{x}{y}\right)$; but this leads to no useful conclusion, because, whatever be the equation of the curve, by putting $u \cdot y = x$, that equation may be transformed into another equation between y and u, whence we may derive $\frac{1}{y} = \mathrm{F}(u) = \mathrm{F}\left(\frac{x}{y}\right)$. The question must therefore be resolved by other considerations. It requires that the two following conditions be fulfilled: First, that for every value of $\frac{x}{y}$, or u, which is always a positive quantity

tity

tity, there shall be two values of $\frac{1}{y}$, since MN must cut the curve in two points. Secondly, that $-\frac{du}{d\frac{1}{y}}$, or $\frac{-du}{d\text{F}(u)}$, (which is the value of PE) shall be of the same value with opposite signs, the two values of $\frac{1}{y}$. The first of these conditions requires that we suppose $\frac{1}{y} = \text{U}' \pm \sqrt{(\text{U})}$, U and U′ being functions of u, which value, on account of the double sign, will give two values of $\frac{1}{y}$ for every value of u; but the second condition will not be fulfilled unless a constant quantity be written in the place of U′. Hence the curves that satisfy the question must fulfil these two conditions.

$$\frac{1}{y} = \text{A} \pm \sqrt{(\text{U})}$$
$$x = y \cdot u.$$

Where U is any function of u.

If we put $\text{U} = \text{B} + \text{C}u + \text{D}u^2 = \text{B} + \text{C}\frac{x}{y} + \text{D}\frac{x^2}{y^2}$, the curve will be defined by this equation

$$\frac{1}{y} = \text{A} \pm \sqrt{(\text{B} + \text{C}\frac{x}{y} + \text{D}\frac{x^2}{y^2})},$$

which includes all the conic sections, and leaves the point P indeterminate or arbitrary.

If $\text{U} = \text{B} + \text{C}\frac{x}{y} + \text{D}\frac{x^2}{y^2} + \text{E}\frac{x^3}{y^3}$, then

$\frac{1}{y} = \text{A} \pm \sqrt{(\text{B} + \text{C}\frac{x}{y} + \text{D}\frac{x^2}{y^2} + \text{E}\frac{x^3}{y^3})}$, which includes all the lines of the third order that satisfy the question.

Solutions were also received from Messrs. Harvey *and* Watts *and* Mr. Phelan.

XI. QUESTION 301, *by* H.

Supposing a particle of matter at rest any where upon the surface of a solid of revolution which revolves about its axis in a given time, and that the particle is attracted towards the centre with a force which is as some determinate function of its distance from the centre: What ought to be the form of the solid?

SOLUTION, *by* G. V.

Let C (fig. 90, pl. 3,) be the centre of the solid, PC its axis of rotation, PAQ a section of the solid by a plane passing through its axis, CQ a perpendicular to the axis meeting the section in Q, and A, A′ any two points in the section PAQ, indefinitely near to one another. Join CA, CA′, draw AB perpendicular to CP, and A′F, A′D perpendicular to CA and AB, respectively.

Put CQ $= a$, CB $= x$, BA $= y$, CA $= r$; then AD $= -dy$, and AF $= -dr$.

Let fr denote the function of r which is proportional to the force tending to the centre, and let g denote the absolute value of that force at Q and c the measure of the centrifugal force at the same point: then the force at A tending to the centre will be $g \times \frac{fr}{fa}$, and the centrifugal force acting in the direction BA will be $\frac{cy}{a}$.

The particle at A is kept in equilibrio by three forces, viz. the force $g\frac{fr}{fa}$ acting in the direction AC, the force $\frac{cy}{a}$ acting in the direction BA, and the reaction of the solid. And by the principles of statics any two of these are to one another as the sines of the angles their directions make with that of the third: therefore, observing that AA′F and AA′D are the angles which the lines AC, AB respectively make with a perpendicular to the curve at A,

$$g\frac{fr}{fa} : \frac{cy}{a} :: (\sin \text{AA}'\text{D} : \sin \text{AA}'\text{F} :: \text{AD} : \text{AF} ::)\ dy : dr;$$

and hence

$$frdr = \frac{fa}{a} \cdot \frac{c}{g}\, ydy.$$

Let $\text{F}r$ be the primitive function of fr. that is, let $fr = \frac{d\text{F}r}{dr}$; then similarly $fa = \frac{d\text{F}a}{da}$; let the values of fr and fa be substituted in the fluxional equation, and it becomes

$$d\text{F}r = \frac{d\text{F}a}{ada} \cdot \frac{c}{g}\, ydy;$$

and hence, taking the fluents,

$$\text{F}r = \frac{d\text{F}a}{2ada} \cdot \frac{c}{g}\, y^2 + Const.$$

To determine the value of the constant quantity, let $x = 0$, then, because in this case $y = r = a$, the equation becomes

$$\mathrm{F}a = \frac{d\mathrm{F}a}{2ada} \cdot \frac{c}{g} a^2 + Const.$$

Let the value of the constant deduced from this equation be substituted in the former, and it becomes

$$\mathrm{F}r = \mathrm{F}a - \frac{d\mathrm{F}a}{2ada} \cdot \frac{c}{g} (a^2 - y^2).$$

This expresses generally the nature of all solids having the property required by the question.

Let us now consider some particular cases, and first, let us suppose that the force is the same at all distances from the centre. Then $fr = r^0 = 1$, and $\mathrm{F}r = r$, therefore $\mathrm{F}a = a$, and $\frac{d\mathrm{F}a}{2ada} = \frac{1}{2a}$. Hence

$$r = a - \frac{1}{2a} \cdot \frac{c}{g} (a^2 - y^3),$$

or, squaring, and putting $x^2 + y^2$ for r^2,

$$x^2 + y^2 = \left\{ a - \frac{1}{2a} \cdot \frac{c}{g} (a^2 - y^2) \right\}^2.$$

This equation shews the section of the solid to be a line of the fourth order.

Next, let us suppose that the force tending to the centre is directly proportional to the distance; here $fr = r$, and $\mathrm{F}r = \frac{1}{2} r^2$, $\mathrm{F}a = \frac{1}{2} a^2$, and $\frac{d\mathrm{F}a}{2ada} = \frac{1}{2}$. Therefore, the equation of the curve is

$$r^2 = a^2 - \frac{c}{g} (a^2 - y^2),$$

which, by putting $x^2 + y^2$ for r^2, may also be expressed thus,

$$\frac{g}{g - c} x^2 = a^2 - y^2.$$

Hence in this case the solid appears to be an oblate spheroid having its greater semi-axis $= a$ and lesser $= \sqrt{\frac{g - c}{g}} \cdot a$.

Lastly, without assigning any particular law to the force tending to the centre, let us suppose that the centrifugal force c is small in comparison to the force g; then, resuming the equation

$$\mathrm{F}r = \mathrm{F}a \frac{d\mathrm{F}a}{2ada} \cdot \frac{c}{g} (a^2 - y^2),$$

let $\mathrm{F}r$ be now regarded as a function of r^2, and expressed thus, $\varphi(r^2)$, so that, similarly, $\mathrm{F}a = \varphi(a^2)$, and it becomes

$$\phi(r^2) = \phi(a^2) - \frac{d\phi(a^2)}{d(a^2)} \cdot \frac{c}{g}(a^2 - y^2).$$

Now the second member of the equation is manifestly the two first terms of the developement of the function $\phi\left\{a^2 - \frac{c}{g}(a^2 - y^2)\right\}$, and these, when $\frac{c}{g}$ is a small fraction, and consequently $\frac{c}{g}(a^2 - y^2)$ also a small quantity, will be nearly equal to the whole function, therefore,

$$\phi(r^2) = \phi\left\{a^2 - \frac{c}{g}(a^2 - y^2)\right\} \text{ nearly,}$$

$$\text{and } r^2 = a^2 - \frac{c}{g}(a^2 - y^2) \text{ nearly.}$$

Hence putting $x^2 + y^2$ for r^2, we get $\frac{g}{g - c}x^2 = a^2 - y^2$; but $\frac{g}{g - c} = \frac{g^2}{(g - \frac{1}{2}c)^2}$ nearly, because by hypothesis c is small in respect of g; therefore,

$$\frac{g^2}{(g - \frac{1}{2}c)^2}x^2 = a^2 - y^2.$$

The solid is therefore nearly an oblate spheroid whose greater axis is a and lesser axis $\frac{2g - c}{2g}a$; and this will be true, whatever be the law of the force tending to the centre.

If the solid be supposed equal in magnitude to the earth, and to revolve in the same time, and have the force of gravity at its equator the same; then $g : c :: 289 : 1$, therefore the ratio of its axes, or $2g : 2g - c$ will be that of $578 : 577$.

Mr. Phelan *sent a solution to this question*

XII. QUESTION 302, *by* G. V.

Theorem. The square of any prime number of the form $4n + 1$ is of the form $a^2 + 25b^2$.

First Solution, *by Mr.* W. Wallace.

I shall extend the proposition, and demonstrate that the square of any prime number of the form $4n + 1$ is either of the form $9a^2 + 25b^2$, or of the form $a^2 + 9 \times 25b^2$.

Every prime number of the form $4n + 1$ is the sum of two squares; this is one of Fermat's theorems, as is generally known; every such number will therefore be of the form

$p^2 + q^2$, and its square of the form $(p^2 + q^2)^2$, an expression equivalent to

$$(p^2 - q^2)^2 + (2pq)^2.$$

It is shewn by Euler in his Algebra (Part 2. Art. 65.) that every square number is either of the form $9m^2$, or of the form $3m + 1$. Now one or other of the two square numbers p^2, q^2 is of the form $9m^2$, or they are both of the form $3m + 1$; in the former case $2pq$ will be of the form $3b$, and in the latter, $p^2 - q^2$ will be of the form $3b$; so that in both cases $(p^2 - q^2)^2 + (2pq)^2$ will be of the form $a^2 + 9b^2$.

Again, Euler also shews (Part 2. Art. 72.) that every square number has some one of these forms, $25m^2$, $5m + 1$, $5m + 4$. Now one or other of the two numbers is of the form $25m^2$; or else they are either both of the form $5m + 1$, or both of the form $5m + 4$, for one cannot be of the form $5m + 1$, and the other of the form $5m + 4$, because then, $p^2 + q^2$ would be of the form $5(m + m' + 1)$, that is of the form $5m''$, which does not agree with the hypothesis of $p^2 + q^2$, or $4n + 1$, being a prime number. If p^2 or q^2 be of the form $25m^2$, then $2pq$ will be of the form $5b$; and if p^2 and q^2 be both of the form $5m + 1$, or else both of the form $5m + 4$, then, in either case, $p^2 - q^2$ will be of the form $5b$; so that in all possible cases, $(p^2 - q^2)^2 + (2pq)^2$ will be of the form $a^2 + 25b^2$.

As it appears that the square of a prime number of the form $4n + 1$ has at the same time the two forms $a^2 + 9b^2$, and $a^2 + 25b^2$; and as the numbers 9 and 25 are prime to one another, it must necessarily have one of the two forms $9a^2 + 25b^2$, $a^2 + 9 \times 25b^2$, as expressed in the general proposition.

EXAMPLES.

$$\begin{array}{ll} 5^2 = 9 \times 0 + 25 \times 1, & 29^2 = 9 \times 7^2 + 25 \times 4^2 \\ 13^2 = 9 \times 4^2 + 25 \times 1, & 37^2 = 9 \times 4^2 + 25 \times 7^2 \\ 17^2 = 8^2 + 9 \times 25 + 1, & 41^2 = 9 \times 3^2 + 25 \times 8^2 \end{array}$$

COROLLARY. The fourth power of any prime number of the form $4n + 1$ is of the form $a^2 + 900b^2$. For it appears from the proposition that it must be of one of these two forms $(9a^2 - 25b^2)^2 + (30ab)^2$, $(a^2 - 225b^2)^2 + (30ab)^2$ both which are included in the form $a'^2 + 900b'^2$.

SECOND SOLUTION, *by* A. B.

If p be a prime number of the form $4n + 1$ it will consist of two squares, one odd and one even; that is $p = (2x)^2 + (2y + 1)^2$; and $p^2 = \left\{(2x)^2 - (2y + 1)^2\right\}^2 + 16x^2 (2y + 1)^2$.

Now all even numbers come under one or other of these forms, viz.

$10z$
$10z + 2$
$10z - 2$
$10z + 4$
$10z - 4$

And all odd numbers come under one or other of these forms, viz.

$5u$
$10u + 1$
$10u - 1$
$10u - 3$
$10u + 3$

The proposition is obvious when $2x = 10z$ or $2y + 1 = 5u$. But these forms, viz.

$$(10z \pm 2)^2 + (10u \pm 1)^2$$
$$(10z \pm 4)^2 + (10u \pm 3)^2$$

cannot be prime numbers because they are divisible by 5 : the only remaining forms are these, viz.

$$(10z \pm 2)^2 \pm (10u \pm 3)^2$$
$$(10z \pm 4)^2 \pm (10u \pm 1)^2.$$

But if $2x = 10z \pm 2$ and $2y + 1 = 10u \pm 3$; or if $2x = 10z \pm 4$, and $2y + 1 = 10u \pm 1$; in every case, $(2x)^2 - (2y+1)^2$ will be divisible by 5, and its square will be divisible by 25.

THIRD SOLUTION, *by* JULIUS.

Every prime number N of the form $4n + 1$, is the sum of two squares, that is, $N = x^2 + y^2$. (*Barlow's Theory of Numbers*, Art. 111.) And consequently

$$N^2 = (x^2 + y^2)(x^2 + y^2) = (x^2 - y^2)^2 + (2xy)^2,$$

or making $\begin{cases} (x^2 - y^2)^2 = x'^2 \\ (2xy)^2 = y'^2 \end{cases}$

we have $N^2 = x'^2 + y'^2$.

But when the sum of two squares is equal to a square, one of the three is divisible by 25, or is of the form $25a^2$, Art. 43 of the abovementioned work.

Now since N is a prime number it cannot be divisible by 25 except in the single case of $N = 5$; therefore it must be one of the other squares x'^2 or y'^2 that is of the form $25a^2$, consequently

$$N^2 = x^2 + 25a^2, \text{ or } b^2 + 25a^2.$$

Mr. CAVILL *sent a solution to this question.*

XIII. QUESTION 303, *by Mr.* CUNLIFFE.

Suppose two projectiles thrown from the same place with equal velocities, in the same vertical plane; to determine the angles of elevation so as to include the greatest space within the curves described in their flight, the horizontal range being the same in both.

FIRST SOLUTION, *by Messrs.* HARVEY *and* WATTS.

Let AP, (fig. 91. pl. 3.) perpendicular to the horizontal line AH, represent the altitude due to the projectile velocity: upon AP as a diameter describe a semi-circle, and draw DEQ parallel to AP, cutting the semi-circle in E and Q; join AE, AQ and draw EF, QR parallel to AH meeting AP in F and R.

Then it is well known that two projectiles delivered in the directions AE, AQ, with the velocities due to the altitude AP, will have the same horizontal range, viz. 4AD: and the respective heights or abscissæ of the parabolic arcs described in the flight will be equal to AF and AR *. Hence $\frac{2}{3} \times 4AD \times (AR - AF) = \frac{8}{3} AD \times FR$ will express the space included by the two curves, which is to be a maximum by the question; therefore $AD \times FR = FE \times FR$ is to be a maximum, that is, the rectangle FEQR inscribed in the semi-circle AEQP is to be a maximum.

Now, it is well known that a square is the greatest rectangle that can be inscribed in a circle, therefore when the rectangle FEQR is a maximum, it will be half a square inscribed in the whole circle, that is $2FE = EQ$, or 2 arc AE = 2 arc PQ = arc EQ. Consequently $\angle AQD = \angle DAE = 22\frac{1}{2}^\circ$, and $\angle DAQ = 67\frac{1}{2}^\circ$.

SECOND SOLUTION, *by Mr.* JOHN CAVILL, *Beighton.*

It is well known that if two projectiles are delivered with equal velocities, and the angle of elevation of the one is the complement of the angle of elevation of the other, then these two projectiles will have equal horizontal ranges. Having made this remark the solution of the question may be as follows.

Let d, denote the velocity per second with which the two projectiles are delivered, and put $g = 16\frac{1}{12}$ feet. Also, let x denote the sine of the angle of elevation of the lower projectile radius 1, then $\sqrt{(1 - x^2)}$ will express the sine of the elevation of the other. And by the doctrine of projectiles in a non-resisting medium $(d^2x \div g)\ \sqrt{(1 - x^2)}$ will express the horizontal range of both projectiles, or the common double ordinate of the two parabolic arcs described by the projectiles. Again by the doctrine of projectiles, $(d^2 \div 4g)\ (1 - x^2)$ will express the altitude or abscissa of the greater arc; and $(d^2x^2 \div 4g)$ that of the less or lower; and therefore $(d^2 \div 4g)\ (1 - 2x^2)$ will express the difference of the two heights.

---

* Dalby's Course of Mathematics, vol. 2, page 326, art. 329.

And hence, by the known principles of mensuration,

$$\frac{2}{3} \times \frac{d^4x}{4g^2}\sqrt{(1-x^2)} \times (1-2x^2) = \frac{d^4}{24g^2} \times 4x\sqrt{(1-x^2)} \times (1-2x^2)$$

will express the area included between the two curves.

Now let A, denote a circular arc radius 1, and sine x; then from the known principles of trigonometry we shall have, sine $4\text{A} = 4x\ \sqrt{(1-x^2)} \times (1-2x^2)$, and by means of this the foregoing expression for the area included between the two curves becomes $(d^4 \div 24g^2)$ sin 4A, which is to be a maximum by the question, therefore sin 4A must be a maximum.

Now when sin 4A is a maximum 4A will be an arc of 90°, because the sine of 90° is the greatest sine; therefore A is an arc of 22½° which is the elevation of the lower projectile, and its complement 67½° is the elevation of the higher.

THIRD SOLUTION, *by Messrs.* WILLIAMS, *and* ERATOSTHENES.

Let θ, φ be the angles made by the directions of projection with the horizon; then by the principles of projectiles $\theta = 90° - \varphi$, or $\sin\theta = \cos\varphi$; and by the problem, the horizontal range multiplied by the difference of altitude is a maximum. Let $g = 16\frac{1}{12}$ feet, and $v =$ the velocity of projection; then the horizontal range $= (v^2 \div 2g) \sin 2\theta$, and the altitudes $= (v^2 \div 4g) \sin^2\theta$ and $v^2 \div 4g) \sin^2\varphi$,

$$\therefore (v^4 \div 8g^2) \sin 2\theta\ (\sin^2\theta - \sin^2\varphi),$$

or, $\sin\theta \cos\theta\ (\sin^2\theta - \sin^2\varphi)$ is a maximum;

$$\therefore (3 \sin^2\theta \cos^2\theta - \sin^4\theta)\, d\theta - (\cos^4\theta - 3\cos^2\theta \sin^2\theta)\, d\theta = 0,$$

or, $6 \sin^2\theta \cos^2\theta - \sin^4\theta - \cos^4\theta = 0$,

$$\therefore \sin 4\theta - \sin^2\theta = -\tfrac{1}{8},$$

and $\sin\theta = \sqrt{(\frac{1}{2} + \sqrt{\frac{1}{8}})} = \cdot 92394 =$ the sine of 67° 30′ nearly, $\therefore \varphi = 90° - \theta = 22° 30'$.

XIV. QUESTION 304, *by* EDINBURGENSIS.

Find the fluxional equation of the *curve of pursuit* when the body pursued moves in any curve of a given nature.

SOLUTION, *by* ALIQUIS.

Let IA (fig. 92. pl. 3.) be the given curve in which the pursued body moves with the uniform velocity m, and NB the required curve in which the pursuing body moves with the velocity n. Let A and B be any cotemporary positions of the bodies, and a, b their position after an indefinitely short interval; then $\mathrm{A}a : \mathrm{B}b :: m : n$, or $\mathrm{A}a = (m \div n)\, \mathrm{B}b$. Let the curves be referred to the straight line KEN, given by position, as an axis; and the abscissæ have their origin at the point N. Draw AK and BE perpendicular to KN, and produce AB to meet the axis in F. Put $\mathrm{NE} = x$, $\mathrm{BE} = y$, $\mathrm{NK} = w$, and $\mathrm{AK} = v$; then $\mathrm{B}b = \sqrt{(\dot{x}^2 + \dot{y}^2)}$, and $\mathrm{A}a = \sqrt{(\dot{w}^2 + \dot{v}^2)}$, or because the curve IA is of a given nature, v is a given function of w, therefore Aa may be denoted by $\dot{w}\sqrt{(f.w)}$, and we shall have $\dot{w}\sqrt{(f.w)} = (m \div n) \times \sqrt{(\dot{x}^2 + \dot{y}^2)}$.

Now since the pursuing body makes always directly towards the other, the line AB joining the bodies, will be a tangent to the curve at B, and the subtangent $\mathrm{EF} = y\frac{\dot{x}}{\dot{y}} = yp$ (putting $\frac{\dot{x}}{\dot{y}} = p$); also by similar triangles EF : EB :: FK : AK; or, in symbols, $yp : y :: w - x + yp : v$; therefore $v = (w - x) \div p + y$; and substituting for v its value, in terms of w, and reducing the equation, we shall obtain, when it is possible, w in terms of x, y and p; or $w = \mathrm{F}(x, y, p)$; hence $f.w = f(\mathrm{F}(x, y, p))$ and $\dot{w} = \dot{\mathrm{F}}(x, y, p)$; these values being substituted in the expression $\dot{w}\sqrt{(f.w)}$, we shall have.

$\dot{\mathrm{F}}(x, y, p)\sqrt{(f(\mathrm{F}(x, y, p)))} = \frac{m}{n}\sqrt{(\dot{x}^2 + \dot{y}^2)}$, for the fluxional equation of the curve sought, in terms of the rectangular co-ordinates and their fluxions, where either $\dot{x}$ or $\dot{y}$ may be supposed to flow uniformly.

When AI is a straight line parallel to the axis KN, y is $= c$, a given line, and $\dot{w} = (m \div n)\sqrt{(\dot{x}^2 + \dot{y}^2)}$; also $v = c = (w - x) \div p + y$, therefore $w = \mathrm{F}(x, y, p)$, $\dot{w} = \dot{\mathrm{F}}(x, y, p) = (c - y)\dot{p} - p\dot{y} + \dot{x} = (c - y)\frac{\ddot{x}}{\dot{y}}$ (supposing $\dot{y}$ to be constant); hence

$(c - y)\frac{\ddot{x}}{\dot{y}} = \frac{m}{n}(\dot{x}^2 + \dot{y}^2)$, or $\frac{\ddot{x}}{\sqrt{(\dot{x}^2 + \dot{y}^2)}} = \frac{m\dot{y}}{n(c - y)}$, the fluxional equation of the curve in this case; agreeing with that at page 516 of Simpson's Fluxions.

The fluxional equation, deduced from the above formula, will, in most cases, be very complex, and we shall in general obtain a much simpler one by seeking the relation between other lines than the rectangular co-ordinates.

Let us take the circle for a particular example. Let C (fig. 93, pl. 3.) be its centre, join CA, CB, and draw CP perpendicular to AB, Ap to ab, and bq to CB. Put $AB = x$, $AP = y$, $CA = r$, $CB = u$, and $CP = p$, then $-\dot{x} = AB - ab = Bb - ap$. But, the tangent to the circle at A being perpendicular to AC, and CP perpendicular to AB, the triangles Aap, ACP will be similar, therefore $ap = Aa \times \frac{p}{r}$, and since $Aa = \frac{m}{n} \times Bb$, $ap = Bb \times \frac{np}{mr}$ and $-\dot{x} = Bb\left(1 - \frac{np}{mr}\right)$. Again, the triangles CBP, bqB are similar, therefore BP $(x-y)$: CB (u) :: Bq $(= CB - Cb = -\dot{u})$: $Bb = \frac{-u\dot{u}}{x-y}$, and consequently $\dot{x} = \frac{u\dot{u}}{x-y} \times \left(1 - \frac{np}{mr}\right)$. But $r^2 - u^2 = y^2 - (x-y)^2 = -x^2 + 2xy$; therefore $u\dot{u} = x\dot{u} - (x\dot{y} + y\dot{x})$, and $\dot{x} = \frac{x\dot{x} - (x\dot{y} + y\dot{x})}{x-y}\left(1 - \frac{np}{mr}\right)$; whence by reduction, $\frac{myr\dot{x}}{np} = x\dot{x} - (xy)^{\cdot}$; or, since $p = \sqrt{(r^2 - y^2)}$, $\frac{mry\dot{x}}{n\sqrt{(r^2 - y^2)}} = x\dot{x} - (xy)^{\cdot}$ the fluxional equation sought.

Eratosthenes *answered it.*

XV. QUESTION 305, *by* L.

To find (from elementary principles) how a ring will move upon a straight rod revolving about one of its ends on an horizontal plane, when, after the first percussive impulse, they are disturbed by no force but their mutual pressure against each other.

SOLUTION, *by Mr.* LOWRY.

Let z denote the distance of the ring from the centre, or the point about which the end of the rod revolves, after any indefinite time t from the commencement of the motion has elapsed; x, and y the rectangular co-ordinates, drawn from the ring, on the straight line from which the motion commences; φ the measure of the angle that the rod or radius vector z, makes with that line reckoned on the circumference of a circle whose radius is 1, v the velocity of the ring in the direction of the rod, u the angular velocity at the distance r from the centre, and P the force by which the ring is urged by the action of the rod in a direction perpendicular to the radius vector: Then this force may be resolved into two others, namely, P $\times \frac{y}{z}$ or P $\sin \varphi$, accelerating the motion of the ring in a direction parallel to, and tending to diminish the ordinate x, and P $\times \frac{x}{z}$ or P $\cos \varphi$, accelerating the motion in a direction parallel to, and tending to augment the ordinate y; therefore by the principles of dynamics

$$\frac{-\ddot{x}}{\dot{t}^2} = \text{P} \sin \varphi,$$

$$\text{and } \frac{\ddot{y}}{\dot{t}^2} = \text{P} \cos \varphi.$$

Let the first of these equations be multiplied by $\cos \varphi$, and the second by $\sin \varphi$; then, by taking the difference, we get

$$\frac{\ddot{x} \cos \varphi + \ddot{y} \sin \varphi}{\dot{t}^2} = 0,$$

and, by adding them together,

$$\frac{-\ddot{x} \cos \varphi + \ddot{y} \sin \varphi}{\dot{t}^2} = \text{P}.$$

But $x = z \cos \varphi$, and $y = z \sin \varphi$; therefore, taking second fluxions, and multiplying the first equation by $\cos \varphi$ and the second by $\sin \varphi$, we get, by addition,

$$\frac{\ddot{x} \cos \varphi + \ddot{y} \sin \varphi}{\dot{t}^2} = \frac{-\ddot{z} + z\dot{\varphi}^2}{\dot{t}^2} = 0,$$

$$\text{or } \frac{\ddot{z}}{\dot{t}^2} = \frac{z\dot{\varphi}^2}{\dot{t}^2}; \ldots\ldots\ldots\ldots (1)$$

and, taking the difference,

$$\frac{-\ddot{x} \cos \varphi + \ddot{y} \sin \varphi}{\dot{t}^2} = \frac{2\dot{z}\dot{\varphi} + z\ddot{\varphi}}{\dot{t}^2},$$

$$\text{or P} = \frac{2\dot{z}\dot{\varphi} + z\ddot{\varphi}}{\dot{t}^2}. \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The first of these equations serves to determine the velocity of the ring, and the second its pressure against the rod. For $\frac{\ddot{z}}{\dot{t}^2} = \frac{\dot{v}}{\dot{t}} = \frac{v\dot{v}}{z}$, and $\frac{\dot{\varphi}^2}{\dot{t}^2} = \frac{u^2}{r^2}$, therefore $v\dot{v} = \frac{u^2 z\dot{z}}{r^2}$,

$$\text{and } v^2 = c^2 + 2\int \frac{u^2 z\dot{z}}{r^2},$$

where c = the value of v when $z = a$.

Again, $\frac{2z\ddot{\varphi}}{\dot{t}\times\dot{t}} = \frac{2vu}{r}$, and $\frac{z\ddot{\varphi}}{\dot{t}^2} = \frac{z\dot{u}}{r\dot{t}} = \frac{vz}{r} \times \frac{\dot{u}}{\dot{z}}$,

$$\text{therefore } P = \frac{v}{r}\left(2u + z\frac{\dot{u}}{\dot{z}}\right).$$

Now let $b =$ the angular velocity when $z = a$, $w =$ the weight of the ring, $w \times m =$ the moment of inertia of the rod, or the sum of the products of each particle by the square of its distance from the centre of motion, which will be $= \frac{1}{3}$ of the square of the length of the rod, multiplied by its weight, when the thickness of the rod is not taken into the account; then the angular velocity, when the ring is at the distance z from the centre, will be $= \frac{(m + a^2)b}{m + z^2}$, or $u = \frac{nb}{m + z^2}$, putting $n = m + a^2$. This value of u being substituted in the expression for the velocity, we have

$$v^2 = c^2 + \frac{2n^2b^2}{r^2} \times \int \frac{z\dot{z}}{(m + z^2)^2},$$

and, taking the correct fluent,

$$v^2 = c^2 + \frac{n^2b^2}{r^2(m + a^2)} - \frac{n^2b^2}{r^2(m + z^2)},$$

or $v = \sqrt{\left(c^2 + \frac{b^2}{r^2}\left(1 - \frac{n^2}{m + z^2}\right)\right)}$, the velocity with which the ring approaches to, or recedes from the centre.

Again $\dot{u} = \frac{-2nbz\dot{z}}{(m + z^2)^2}$, therefore $z\frac{\dot{u}}{\dot{z}} = \frac{-2nbz^2}{(m + z^2)^2}$,

and $2u + z\frac{\dot{u}}{\dot{z}} = \frac{2mnb}{(m + z^2)^2}$, and $P = \frac{v}{r}\left(2u + z\frac{\dot{u}}{\dot{z}}\right) =$

$$\frac{2mnb}{r(m + z^2)^2} \times \sqrt{\left(c^2 + \frac{b^2}{r^2}\left(1 - \frac{n^2}{m + z^2}\right)\right)}.$$

To determine the nature of the curve described by the ring, we have

$$\dot{\varphi} = \frac{u}{r}\dot{t} = \frac{u\dot{z}}{rv} = nb\dot{z} \times \frac{m + z^2}{(r^2c^2 + nb^2)(m + z^2) - n^2b^2},$$

where the fluent is always assignable by elliptic arcs or by logarithms.

XVI. QUESTION 296, *by Mr.* W. WALLACE.

If the intersection of two straight lines which contain a given angle be on a line of any kind given by position, and one of them pass through a given point, what is the nature of the curve to which the other is a tangent?

FIRST SOLUTION, *by Mr.* WALLACE, *the Proposer.*

In resolving this question I shall suppose the given angle a right angle, for the sake of simplicity, and also because every other case may be reduced to this. For let A be a given point, (fig. 94. pl. 3.) MBN a line of any kind given by position, and AB, BC two lines which contain a given angle B, and one of which passes through the given point A. Draw AB′ perpendicular to BC, also draw AP any line given by position, and AP′ another line, so that the angles, BAP, B′AP′ may be equal; then the angle PAP′ will be equal to the angle BAB′, that is to a given angle; therefore AP′ will also be given by position: And since the angle B′AP′ is equal to the angle BAP; and B′A has to BA a given ratio, (for the triangle BAB′ is given in species) it is manifest that the line M′B′N′ which is the *locus* of the point B′ will be similar to the line MBN which is the locus of B; so that when the latter is given, the former is also given.

To proceed now with the solution of the problem; let MBN be the given line (fig. 95. pl. 3.), A the given point, and AB, BC the two lines which are to meet in the line MBN, and form a right angle. Let AB′C be another position of the lines, indefinitely near to the former; then C the intersection of the lines BC, B′C is manifestly a point in the curve to which BC is a tangent.

Take a straight line AX given in position, to which the curves may be referred, as an axis. Draw CD, BE perpendicular to AX, and HBF perpendicular to CD; also, draw B′H perpendicular to FB, let B′A meet BC in O and BF in G, and draw the tangent B′BK meeting the axis in K.

The triangles GBO, GB′H are similar; so also are the triangles B′BG, BKA, and the triangles BOB′ ABC; and lastly, the triangles BCF, BAE; hence,

$$\mathrm{B'H : BO\ (:: B'G : GB) :: BA : AK},$$
$$\mathrm{BO : B'O\ (:: AB : BC) :: BE : BF},$$

therefore, compounding these ratios,

$$\mathrm{B'H : B'O :: BA \times BE : AK \times BF},$$

and $\mathrm{BF} = \dfrac{\mathrm{BE \times BA \times B'O}}{\mathrm{AK \times B'H}}$.

In like manner, from the ratios

$$\mathrm{B'H : BO\ (:: B'G : GB) :: BA : AK},$$
$$\mathrm{BO : B'O\ (:: AB : BC) :: AE : CF},$$

we obtain $\mathrm{B'H : B'O :: BA \times AE : AK \times CF}$, and

$$\mathrm{CF} = \frac{\mathrm{AE \times BA \times B'O}}{\mathrm{AK \times B'H}}.$$

Let us now put AE the abscissa of MBN, the given curve

$= r$; EB its ordinate $= s$; AB the variable radius $= v$, and AK, the segment of the axis between A and the tangent, $= t$. Also, let AD the abscissa of the curve sought be x, and DC its ordinate y; then B'H $= ds$, B'O $= dv$, these symbols being substituted we get

$$\text{BF} = \frac{svdv}{tds}, \quad \text{CF} = \frac{rvdv}{tds};$$

Therefore, from the position of the lines we immediately obtain these equations,

$$x = r - \frac{svdv}{tds} \dots\dots\dots\dots\dots\dots (1).$$

$$y = s + \frac{rvdv}{tds} \dots\dots\dots\dots\dots\dots (2).$$

The nature of the curve furnishes a third equation, by which, three of the four quantities r, s, t, v may be expressed in terms of the remaining one, and this done, we have only to eliminate that quantity from the above two equations, and the result will be the equation of the curve whose nature is required.

We may also express the nature of the curve independently of v and t, by remarking that $v^2 = r^2 + s^2$, from which we get $vdv = rdr + sds$; also that $t = r - \frac{dr}{ds}s$; hence by substitution we have

$$x = \frac{(r^2 - s^2)\, ds - 2rsdr}{rds - sdr} \dots\dots\dots (3),$$

$$y = \frac{(r^2 - s^2)\, dr + 2rsds}{rds - sdr} \dots\dots\dots (4);$$

by these, and the equation of the given curve, we may eliminate r and s, and the resulting equation will express the nature of the curve to which EC is a tangent, as required by the question.

Ex. 1. Let the given line MBN (fig 96. pl. 3.) be a straight line. Let the axis AX cut it right angles in N. Put AN $= c$. In this case we have $s =$ NB, $r =$ AN $= c$ a constant quantity, therefore $dr = 0$; $r^2 - s^2 = c^2 - s^2$; $rs = cs$, and by substitution in formulæ (3) and (4)

$$x = \frac{c^2 - s^2}{c}, \qquad y = 2s.$$

Hence, eliminating s, we have $y^2 = 4c\,(c - x)$, an equation to a parabola of which A is the focus, AX the axis, and $4c$ the parameter.

Ex. 2. Let the given curve MBN be a circle (fig. 95. pl. 3.) whose centre is A, and radius MX. Put AX $= c$, XM $= a$,

$e^2 - a^2 = b^2$. By the nature of the circle, $r^2 + s^2 = 2er - b^2$, hence $ds = \frac{(e-r)dr}{s}$, and by the third and fourth formulæ

$$x = \frac{b^2e - (2e^2 - b^2)r}{b^2 - er}, \quad y = \frac{b^2s}{b^2 - er}.$$

To simplify the equation of the curve, it will be convenient in this case to change the beginning of the abscissæ x from A to X; to effect this we must put $x' = e - x$; then we have

$$x' = \frac{a^2r}{b^2 - er}, \quad y = \frac{b^2s}{b^2 - er},$$

from these equations we get

$$r = \frac{b^2x'}{a^2 + ex'}, \quad s = \frac{a^2y}{a^2 + ex'};$$

substituting now these values of r and s in the equation $r^2 + s^2 = 2er - b^2$, and reducing, we find

$$a^2y^2 = b^2(x'^2 - a^2)$$

for the equation of the curve, which shews it to be a conic section whose semi-axes are a and b.

SECOND SOLUTION, *by Mr.* LOWRY.

Let HCG (fig. 97, pl. 3.) be the given curve, A the given point, and KE the curve required, such, that the lines AC, CE being drawn to any point C in the curve to include a given angle, the line CE shall be a tangent to the curve KE. Let the straight line AX, drawn through the given point, be taken for the axis of the curves, and instead of the ordinates being at right angles to this axis let them make with it an angle equal to the given one; that is, let the angles ABC, FDE, made by the ordinates CB, DE, be equal to the given angle ACE.

Let AB, the abscissa of the given curve, be $= u$: BC, its ordinate, $= v$; AD, the abscissa of the required curve, $= x$; DE, its ordinate, $= y$; and the cosine of the given angle $= a$. Then because of the equal angles ABC, ACE, the triangles ACB, ACF are similar; therefore AB : AC :: AC : AF $= \frac{\text{AC}^2}{\text{AB}} = \frac{u^2 + v^2 - 2avu}{u}$, and FD $= \frac{u^2 + v^2 - 2avu}{u} - x$. Again, the triangles ABC, EDF are similar; therefore BC : AB :: DF : DE

$$= \frac{u}{v} \times \text{DF} = \frac{u^2 + v^2 - 2avu - ux}{v} = y;$$

$$\text{or } u^2 + v^2 - 2avu - ux - vy = 0 \ldots.(1),$$

which is an equation to the straight line CE, in terms of the co-ordinates x and y, and the arbitrary quantities u and v (given functions of one another), and corresponds to the general equation $f(x,\ y,\ a) = 0$, in the solution to question 279; this question being only a particular example to the general problem which is there resolved; therefore taking fluxions, considering u and v alone to vary, as directed in that solution, we shall have

$$\frac{\dot{v}}{\dot{u}} = \frac{2u - 2av + x}{2au - 2v + y} \ldots\ldots\ldots\ldots\ldots (2).$$

Now from these equations, and that of the given curve, we can exterminate u and v, and the result will be the equation of the curve required, in terms of x and y, independent of u or v.

When the given angle is a right one, the co-ordinates are rectangular, and $a = 0$; the above equations then become

$$u^2 + v^2 - ux - vy = 0 \ldots\ldots\ldots (3)$$

$$\text{and } \frac{\dot{v}}{\dot{u}} = \frac{2u - x}{y - 2v}; \ldots\ldots\ldots\ldots\ldots (4)$$

from whence we get by elimination,

$$x = \frac{(u^2 - v^2)\,\dot{v} - 2uv\dot{u}}{u\dot{v} - v\dot{u}} \ldots\ldots\ldots (5)$$

$$y = \frac{(a^2 - v^2)\dot{u} + 2uv\dot{v}}{u\dot{v} - v\dot{u}} \ldots\ldots\ldots (6)$$

For a particular example, let us suppose the given curve to be a circle, and the angle a right one. Through the given point A (fig. 98. pl. 3.) draw the diameter HOG; put $r =$ the radius OG, and $a =$ the distance of the given point from the centre: then, by the nature of the circle, $r^2 = v^2 + (a - u)^2$. Now it is evident from the above formulæ, that there are several ways in which u and v may be exterminated; we may either make use of equation (3) and that which expresses the nature of the given curve; or, of equations (5) and (6) and that of the given curve: in the present case it will perhaps be as convenient to make use of equation (3). Since $v^2 = r^2 - (a - u)^2$, and $v = \sqrt{(r^2 - (a - u)^2)}$; if these values be substituted in equation (3) it becomes

$$r^2 - a^2 - 2au - ux - y\sqrt{(r^2 (a - u)^2)} = 0;$$

or, when cleared of the radical, and the terms properly arranged,

$$(r^2 - a^2 - y^2)(r^2 - a^2) + (2(r^2 - a^2)(2a - x) - 2ay^4)u + ((2a - x)^2 + y^2)u^2 = 0.$$

Taking the fluxion, supposing the arbitrary quantity u only to vary, we have $u = -\dfrac{(r^2 - a^2)(2a - x - ay^2)}{(2a - x)^2 + y^4}$, and substituting this value of u, in the above equation, we have, after proper reduction, $r^2y^2 = (r^2 - a^2 + 2ax - x^2)(r^2 - a^2)$, an equation to an ellipse or hyperbola, according as the given point is within or without the circle: the point A being the focus, and the line HG the transverse axis of the curve.

Harvey *and* Watts *of Plymouth answered this question.*

XVII. QUESTION 307, *by Mr.* LOWRY.

Let AEC, BFC and EF be three straight lines given by position: From the given points A, B let straight lines be inflected to any point I in EF, meeting AC in P, and BC in Q, and let P, Q be joined. Required the nature of the curve to which PQ is always a tangent?

SOLUTION, *by Mr.* LOWRY.

Draw AD and BH (fig. 99. pl, 3.) parallel to EF, meeting CB in D and CA in H, let O be one of the points of contact, and draw NO parallel to CB. Put EH $= a$, EF $= b$, HB $= c$, AD $= d$, FD $= e$, CH $= m$, CD $= n$, PH $= u$, QD $= v$, HN $= x$, and NO $= y$. Then, by similar triangles

$$\text{PH}\ (u) : \text{HB}\ (c) :: \text{PE}\ (u - a) : \text{EI} = (c \div u)\,(u - a),$$

$$\text{QD}\ (v) : \text{AD}\ (d) :: \text{QF}\ (v - e) : \text{FI} = (d \div v)\,(v - e);$$

therefore $\dfrac{c(u-a)}{u} + \dfrac{d(v-e)}{v} = \text{EF} = b,$

and by reduction $(c + d - b)\,uv - cav - edu = 0,$
or, putting $r = ca \div (c + d - b)$ and $s = ed \div (c + d - b)$ to simplify the expressions, then $v = \dfrac{su}{u - r}.$

Again, the triangles PNO, PCQ are similar, therefore

$$\text{PC}\ (m - u) : \text{CQ}\ \left(n - v = n - \frac{su}{u-r}\right) :: \text{PN}\ (x - u) : \text{NO}\ (y),$$

or, $my - nx - uy + nu = (u - x)\dfrac{su}{u - r};$

and, by reduction,

$$rnx - rmy + (my - nx + ry - nr + sx)u + (s + y - n)u^2 = 0,$$

which is an equation to the straight line PQ in terms of the co-ordinates x and y, and the arbitrary quantity u, and corresponds to the general equation $f(x, y, a) = 0$, alluded to in the solution to the preceding question; therefore to exterminate u, take the fluxion, supposing that quantity only to vary,

and there results $u = -\dfrac{my - nx + ry - nr + sx}{2(s + y - n)}$, which being substituted for u in the above equation we have

$$4(rnx - rmy)(s + y - n) - (my - nx + ry - nr + sx)^2 = 0,$$

for the equation of the curve sought, namely, that of an ellipse inscribed in the quadrilateral APQB, and touching the lines AE, BC at the points E and F.

The question may also be resolved rather differently, as follows:

The several substitutions being retained, except u and v which are not wanted, the subtangent PN is $= y\frac{\dot{x}}{\dot{y}}$, $\mathrm{HP} = x - y\frac{\dot{x}}{\dot{y}}$, $\mathrm{DQ} = \frac{s \times \mathrm{HP}}{\mathrm{HP} - r} = s\left(x - y\frac{\dot{x}}{\dot{y}}\right) \div \left(x - y\frac{\dot{x}}{\dot{y}} - r\right)$, $\mathrm{CQ} = n - \mathrm{DQ}$ $= \left(nx - ny\frac{\dot{x}}{\dot{y}} - nr - sx + sy\frac{\dot{x}}{\dot{y}}\right) \div \left(x - y\frac{\dot{x}}{\dot{y}} - r\right)$, and by similar triangles

$\mathrm{PC}\left(m - x + y\frac{\dot{x}}{\dot{y}}\right) : \mathrm{CQ} :: \mathrm{PN}\left(y\frac{\dot{x}}{\dot{y}}\right) : \mathrm{NO}\ (y)$; therefore substituting for CQ its value, and multiplying extremes and means, we have,

$$\left(m - x + y\frac{\dot{x}}{\dot{y}}\right)\left(x - y\frac{\dot{x}}{\dot{y}} - r\right) = \left(nx - nr - sx + (s-n)y\frac{\dot{x}}{\dot{y}}\right)\frac{\dot{x}}{\dot{y}},$$

or, by reduction and a proper arrangement of the terms

$$(mx - x^2 - mr + rx) - (my - nr - sx + nx + ry - 2xy)\frac{\dot{x}}{\dot{y}} + y(-n + s + y)\frac{\dot{x}^2}{\dot{y}^2} = 0,$$

for the fluxional equation of the curve.

To deduce the integral, which expresses the nature of the curve, from the fluxional equation, in problems of this kind, mathematicians generally take the fluxion, supposing either $\dot{x}$ or $\dot{y}$ to be constant (this supposition being always admissible), by which means they obtain the value of $\frac{\dot{x}}{\dot{y}}$ in terms of the co-ordinates; and this value being substituted in the original equation, the result is the integral sought. But this integral may be found by a much easier process, for it is unnecessary to consider any of the quantities as variable, except $\left(\frac{\dot{x}}{\dot{y}}\right)$, and the fluxion being taken on this supposition, we immediately obtain the value of $\frac{\dot{x}}{\dot{y}}$, which being substituted as before, we have the equation of the curve sought.

Thus, in the above equation, if the fluxion be taken on the supposition that $\left(\frac{\dot{x}}{\dot{y}}\right)$ only is variable, we have

$$-(my - nr - sx + nx + ry - 2xy)\left(\frac{\dot{x}}{\dot{y}}\right)^{\cdot} + 2y(-n + s + y)\frac{\dot{x}}{\dot{y}}\left(\frac{\dot{x}}{\dot{y}}\right)^{\cdot} = 0;$$

therefore $\frac{\dot{x}}{\dot{y}} = \frac{my - nr - sx + nx + ry - 2xy}{2y(-n + s + y)}$;

and substituting this value of $\frac{\dot{x}}{\dot{y}}$ in the fluxional equation of the curve, we have

$$(mx - x^2 - mr + rx) - \frac{(my - nr - sx + nx + ry - 2xy)^2}{4y(-n + s + y)} = 0,$$

which, by considering that $my - nr - sx + nx + ry - 2xy$ is $= (my - nx + ry - nr + sx) - 2x(-n + s + y)$, reduces immediately to

$$4(rnx - rmy)(s + y - n) - (my - nx + ry - nr + sx)^2 = 0,$$

the same as before.

In general, if the fluxional equation of the curve be of the form $P + Q\frac{\dot{x}}{\dot{y}} + S\frac{\dot{x}^2}{\dot{y}^2} = 0$, where P, Q and S are functions of the co-ordinates x and y, and constant quantities; then $Q\left(\frac{\dot{x}}{\dot{y}}\right)^{\cdot} + 2S\frac{\dot{x}}{\dot{y}}\left(\frac{\dot{x}}{\dot{y}}\right)^{\cdot}$ being $= 0$, we have $\frac{\dot{x}}{\dot{y}} = -\frac{Q}{2S}$;

and, by substitution, $P - \frac{Q^2}{2S} + \frac{Q^2}{4S} = 0$;

or, $4PS - Q^2 = 0$, the general equation of the curve.

To apply this formula to a particular example, let A and B be given points, in the straight lines AC, DB given by position; required the nature of the curve to which PQ is a tangent when the rectangle AP . BQ is a constant quantity c^2.

First, when the lines are parallel (fig. 100. pl. 3.), let O be a point of contact, and draw ON parallel to AB, meeting BQ in M. Put AN $= x$, ON $= y$, and AB $= a$; then NP $= y\frac{\dot{x}}{\dot{y}}$, AP $= x - y\frac{\dot{x}}{\dot{y}}$, MQ $= (a - y)\frac{\dot{x}}{\dot{y}}$, and BQ $= x + (a - y)\frac{\dot{x}}{\dot{y}}$;

hence AP · BQ $= \left(x - y\frac{\dot{x}}{\dot{y}}\right)\left(x + (a - y)\frac{\dot{x}}{\dot{y}}\right) = x^2 + (ax - 2xy)\frac{\dot{x}}{\dot{y}} + (y^2 - ay)\frac{\dot{x}^2}{\dot{y}^2} = c^2$; therefore $P = x^2 - c^2$, $Q = ax - 2xy$, $S = y^2 - ay$, and, by the formula, $4(x^2 - c^2)(y^2 - ay) = (ax - 2xy)^2$; or, by reduction, $x^2 = \frac{4c^2}{a^2}(ay - y^2)$, which is an equation to an ellipse when the curve falls between the given lines, and to an hyperbola when it falls without, a and c being semi-conjugate diameters of the section.

Secondly, when the lines are not parallel, but intersect, when produced, in C (fig. 99. pl. 3.). Put $AC = a$, $BC = b$, $AN = x$ and $NO = y$; then $NP = y\frac{\dot x}{\dot y}$, $AP = x - y\frac{\dot x}{\dot y}$, $CQ = y +$ $(a -$ [illegible] $\frac{\dot y}{\dot x}$; and $BQ = b - y - (a - x)\frac{\dot y}{\dot x}$; hence $AP \cdot BQ =$ $\left(x - y\frac{\dot x}{\dot y}\right)\left(b - y - (a - x)\frac{\dot y}{\dot x}\right) = c^2$, or $x^2 - ax + (bx +$ $ay - 2xy - c^2)\frac{\dot x}{\dot y} + (y^2 - by)\frac{\dot x^2}{\dot y^2} = 0$,

therefore $P = x^2 - ax$, $Q = bx + ay - 2xy - c^2$, $S = y^2 - by$, and, by the formula, $4(x^2 - ax)(y^2 - by) = (bx + ay - 2xy - c^2)^2$, or, by reduction, $(bx + ay - c^2)^2 = 4xy(ab - c^2)$, which is an equation to an ellipse, or hyperbola, according as the curve falls within or without the angle ACB.

XVIII. QUESTION 308, *by Mr.* W. WALLACE.

From two given points equally distant from the centre of a given circle draw straight lines to a point in the circumference so that the sum of the fourth powers of these lines may be a given quantity.

FIRST SOLUTION, *by Mr.* JOHN CAVILL.

Let A and B (fig. 101. pl. 3.) be the given points equally distant from the centre O of the given circle ECF, and suppose the lines AC, BC to be drawn to the point C as required, so that the sum of their fourth powers may be a constant quantity.

Bisect the line AB in D, and join OD, meeting the circle in E and F; also draw CP perpendicular to AB, and CI to DF; then because the points A, B are equally distant from the centre O, the line OD will be perpendicular to AB, or parallel to CP, and therefore $DP = CI$.

Now $AC^2 = DC^2 + AD^2 + 2AD \cdot DP$,

and $BC^2 = DC^2 + AD^2 - 2AD \cdot DP$;

therefore $AC^4 + BC^4 = 2(DC^2 + AD^2)^2 + 8AD^2 \cdot DP^2$;

But $DC^2 = OC^2 + OD^2 + 2OD \cdot OI$; therefore, if $2OD \cdot OK$ be taken $= OC^2 + OD^2 + AD^2$, K will be a given point, and $DC^2 + AD^2 = 2OD \cdot IK$; therefore $AC^4 + BC^4 = 8OD^2 \cdot IK^2 + 8AD^2 \cdot DP^2 =$ a constant quantity, $=$ (suppose to) $8OD^2 \cdot s^2$; consequently

$OD^2(IK^2 - s^2) = AD^2 \cdot IC^2$; or $IK^2 - s^2 : IC^2 :: AD^2 : OD^2$.

Take KG and KH, on different sides of K, each equal to S; then $IK^2 - S^2 = GI \cdot IH$, and by the property of the circle

$IC^2 = EI \cdot IF$; therefore, $GI \cdot IH : EI \cdot IF :: AD^2 : OD^2$,

a given ratio; the problem is therefore reduced to a case of determinate section. See Lawson's 4th prob.

SECOND SOLUTION, *by* ALIQUIS.

Let A and B (fig. 102. pl. 3.) be the given points, O the centre of the given circle, and C the point to which the lines AC, BC are to be drawn so that the sum of their fourth powers may be a given quantity. From C let fall the perpendiculars CD, CE on the lines AO, BO produced, and join CO. Then $AC^2 = AO^2 + OC^2 + 2AO \cdot OD$, and $BC^2 = BO^2 + OC^2 + 2BO \cdot OE = AO^2 + OC^2 + 2AO \cdot OE$ (because $OB = AO$ by hypothesis), or if a point P be found, in AO, so that $2OA \cdot OP$ may be $= AO^2 + OC^2$, and OR be taken $= OP$, then $AC^2 = 2AO \cdot PD$, $BC^2 = 2AO \cdot RE$, and $AC^4 + BC^4 = 4AO^2 \times (PD^2 + RE^2) =$ a given quantity; therefore $PD^2 + RE^2$ is a given space $= 2R^2$.

Now because of the right angles at D and E, the points O, D, C, E are in a circle whose diameter is OC, the radius of the given circle; and if OG be drawn to bisect the given angle EOS, formed by the lines AO, BO produced, and meet the circle in G, the line EG will be given: also if GN be drawn perpendicular to OE, RN will be half the sum and EN half the difference of RE and PD, and therefore $PD^2 + RE^2 = 2EN^2 + 2NR^2 = 2EG^2 - 2GN^2 + 2NR^2 = 2R^2$, and consequently $NR^2 - GN^2 = R^2 - EG^2$. But $NR^2 = ON^2 + 2RO \cdot ON + RO^2$, and because of the given angles, NG^2 has to ON^2 a given ratio, or

$NG^2 = \frac{m}{n} ON^2$; therefore

$$ON^2 - \frac{m}{n} ON^2 + 2RO \cdot ON = R^2 - EG^2 - RO^2,$$

$$\text{or, } ON^2 + \frac{2n}{n-m} RO \cdot ON = (R^2 - EG^2 - RO^2) \frac{n}{n-m};$$

therefore the point N may be found by Prob. 18. v. Simpson's Geometry, and the construction will then be obvious.

THIRD SOLUTION, *by* ERATOSTHENES.

Let A and B (fig. 103. pl. 3.) be the given points, and K the required point; draw the diameter EOG and the line KL perpendicular to AB, and KS perpendicular to EC; also join KA, KB, KG and OK: Then, because the points A and B are equally distant from the centre, AG is $= GB$. Also (Euc. 12 and 13, ii.)

$\text{AK}^2 = \text{AG}^2 + \text{GK}^2 + 2\text{AG}\cdot\text{GL}$ and $\text{BK}^2 = \text{AG}^2 + \text{GK}^2 - 2\text{AG}\cdot\text{GL}$, therefore

$$\text{AK}^4+\text{BK}^4=(\text{AG}^2+\text{GK}^2+2\text{AG}\cdot\text{GL})^2+(\text{AG}^2+\text{GK}^2-2\text{AG}\cdot\text{GL})^2$$
$$=2(\text{AG}^2+\text{GK}^2)^2+8\text{AG}^2\cdot\text{GL}^2:$$

But $\text{GK}^2=\text{OG}^2+\text{OK}^2+2\text{OG}\cdot\text{OS}$, and $\text{GL}^2=\text{SK}^2=\text{OK}^2-\text{OS}^2$; therefore

$$\text{AK}^4+\text{BK}^4=2(\text{AG}^2+\text{OG}^2+\text{OK}^2+2\text{OG}\cdot\text{OS})^2+8\text{AG}^2\times(\text{OK}^2-\text{OS}^2)$$
$$=2(\text{AG}^2+\text{OG}^2+\text{OK}^2)^2+8\text{AG}^2\cdot\text{OK}^2+8\text{OG}(\text{AG}^2+\text{OG}^2+\text{OK}^2)\cdot\text{OS}+8(\text{OG}^2-\text{AG}^2)\cdot\text{OS}^2$$

a constant quantity $= \text{V}^4$, and as OS rises only to the second dimension, the equation is easily constructed by the right line and circle.

FOURTH SOLUTION, *by Mr.* CUNLIFFE.

Let A and B (fig. 103. pl. 3.) be the given points, EKD the given circle, and AK, BK the required right lines, drawn to the point K in the circumference: draw the diameter ED perpendicular to AB, the right line joining the given points; draw KS parallel to AB meeting ED in S; also draw the radius OK.

Put $\text{AG}=\text{BG}=a$, $\text{OG}=b$, $\text{OK}=r$, $\text{GL}=\text{KS}=x$, and $\text{OS}=y$: Then

$$\text{AK}^2=(a+x)^2+(b+y)^2=a^2+b^2+2ax+2by+x^2+y^2=a^2+b^2+r^2+2ax+2by$$
$$=s^2+2ax+2by, \text{ by putting } s^2=a^2+b^2+r^2;$$
$$\text{BK}^2=(a-x)^2+(b+y)^2=a^2+b^2+r^2-2ax+2by=s^2-2ax+2by;$$
$$\text{AK}^4=s^4+2s^2(2ax+2by)+4a^2x^2+8abxy+4b^2y^2;$$
$$\text{BK}^4=s^4+2s^2(2by-2ax)+4a^2x^2-8abxy+4b^2y^2;$$
$$\text{AK}^4+\text{BK}^4=2s^4+8s^2by+8a^2x^2+8b^2y^2=2s^4+8s^2by+8a^2(r^2-y^2)+8b^2y^2=$$

$2s^4+8a^2r^2+8s^2by-8a^2y^2+8b^2y^2$, which is to be equal to a given quantity. Now as the highest power of the unknown quantity in this expression does not exceed the square, the problem may be constructed geometrically, that is by the intersection of a right line and circle.

When $b = 0$, or when the right line AB, joining the given points, passes through the centre of the given circle, the expression for the sum of the fourth powers of the lines AK and BK becomes $2s^4-8a^2y^2$, which is very simple. When $a = b$, the said expression becomes $2s^4 + 8a^2r^2 + 8s^2by$, which shews that in that case, the problem may be constructed by the intersection of right lines.

Messrs. Harvey *and* Watts, *of Plymouth, anſwered it.*

XIX. QUESTION 309, *by* AMICUS.

A point and a straight line are given by position in the same vertical plane; a heavy body, urged by the force of gravity, has to descend from the given point, in a circular arc, so as to arrive at the given line in the shortest time possible. What is the length of the arc?

Solution, *by* Amicus, *the Proposer.*

Let P (fig. 104. pl. 3.) be the given point, DE the straight line given by position, and PK the required arc, through which a heavy body will descend in less time than through any other circular arc having its centre on the same straight line and passing through the given point P. Suppose, for greater simplicity, that the centres are in the horizontal line PD, and let *pk* be another arc similar to PK, meeting the line *de*, parallel to DE, in *k*; that is let *pk* be the same part of the quadrant *pb*, that PK is of the quadrant PB; and *pd* the same part of the radius *pa*, that PD is of the radius PA; then, if PK be the arc of quickest descent from P to DE, *pk* will be the arc of quickest descent from *p* to *de*. Suppose now the length of the arc *pk* to be such, that a body would descend through it in the given time t; and that *skn* is the synchronous curve to all the circles that have their centres on the horizontal line *pd*, and pass through *p*; or the curve that cuts them all in such a manner, that the time of descent from the given point to the curve is the same in each circle. Then because *pk* is the arc of quickest descent to the line *de*, it is evident that every point in the synchronous curve except *k*, will fall above the line *de*, therefore *de* will be a tangent to the curve at *k*. Draw the vertical lines *kc*, *pe*, and put $x = ke$, $y = pe$, $z =$ the arc *pk*, $h =$ the space that a body would descend vertically in the given time t, and $g = 32\frac{1}{6}$ feet. Then if the nature of the synchronous curve be expressed by the equation $f(x, y, h) = 0$, we shall have $\frac{\dot{x}}{\dot{y}} = \text{F}(x, y, h)$; but $-\frac{\dot{x}}{\dot{y}}$ is $=$ the tangent of the given angle *ped*, $= a$; therefore $\text{F}(x, y, h) = -a$, and from this equation, and that of the synchronous curve, we can determine the values of x and y in terms of the given quantities h and a, and also the lengths of the arc *pk* and the line *pd*; then it will be as, *pd* : PD :: arc *pk* : arc PK.

The only difficulty now remaining, is, to find the equation of the synchronous curve: This has been done by several authors, as Euler, Bossut, &c. in a manner not much different from the following: By the laws of descending bodies the time of descent in the arc *pk* is $= \int \frac{\dot{z}}{\sqrt{(2gx)}}$, and by the nature of the circle $\dot{z} = \frac{r\dot{x}}{\sqrt{(r^2 - x^2)}}$; therefore

$$\int \frac{r\dot{x}}{\sqrt{2g}\sqrt{(r^2x - x^3)}} = t = 2\sqrt{\left(\frac{h}{2g}\right)},$$ and multiplying by $\sqrt{2g}$,

$\int \frac{r\dot{x}}{\sqrt{(r^2x-x^3)}} = 2\sqrt{h}$; or, putting $P = \frac{r}{\sqrt{(r^2x-x^3)}}$, $\int P\dot{x} = 2\sqrt{h}$, and differencing *de curva curvam*, $P\dot{x} + \dot{r}\int\left(\dot{x}\left(\dot{\overline{\tfrac{P}{r}}}\right)\right) = 0$.

$$\text{Now } \left(\dot{\overline{\tfrac{P}{r}}}\right) = \frac{1}{(\sqrt{r^2x-x^3})} - \frac{r^2x}{(r^2x-x^3)^{\frac{3}{2}}} = \frac{1}{2\sqrt{(r^2x-x^3)}} - \frac{r^2+x^2}{2\sqrt{x}\cdot(r^2-x^2)^{\frac{3}{2}}},$$

$$\dot{x}\left(\dot{\overline{\tfrac{P}{r}}}\right) = \frac{\dot{x}}{2\sqrt{(r^2x-x^3)}} - \frac{(r^2+x^2)\dot{x}}{2\sqrt{x}\,.(r^2-x^2)^{\frac{3}{2}}} = \frac{P\dot{x}}{2r} - \left(\frac{\sqrt{x}}{\sqrt{(r^2-x^2)}}\right)^{\cdot},$$

$$\int\dot{x}\left(\dot{\overline{\tfrac{P}{r}}}\right) = \frac{\sqrt{h}}{r} - \frac{\sqrt{x}}{\sqrt{(r^2-x^2)}},\quad P\dot{x} + \frac{\dot{r}}{r}\sqrt{h} - \dot{r}\frac{\sqrt{x}}{\sqrt{(r^2-x^2)}} = 0;$$

$$\text{therefore}\quad \frac{r\dot{x}}{\sqrt{(rx-x^3)}} + \frac{\dot{r}}{r}\sqrt{h} - \dot{r}\frac{\sqrt{x}}{\sqrt{(r^2-x^2)}} = 0,$$

$$\text{or } \frac{r\dot{x}-x\dot{r}}{\sqrt{(r^2-x^2)}} + \frac{\dot{r}}{r}\sqrt{(hx)} = 0 \ldots\ldots\ldots (1)$$

To exterminate r and its fluxion, we have, by the property of the circle, $2ry - y^2 = x^2$; therefore $r = \frac{y^2+x^2}{2y}$, $\dot{r} = \left(\frac{y^2+x^2}{2y}\right)^{\cdot} = \frac{(y^2-x^2)\dot{y} - 2yx\dot{x}}{2y^2}$; and these values being substituted in equation (1) it becomes

$(y\dot{x}-x\dot{y})(y^2+x^2) - (y^2-x^2)\dot{y} - 2yx\dot{x})\sqrt{(hx)} = 0 \ldots\ldots (2)$,

whence $\frac{\dot{x}}{\dot{y}} = \frac{(y^2+x^2)x + (y^2-x^2)\sqrt{(hx)}}{(y^2+x^2)y + 2yx\sqrt{(hx)}} = F(x, y, h) = -a$.

Now to integrate equation (2), let it be divided by y^2 and put under this form, viz.

$$\left(\frac{x}{y}\right)^{\cdot}(y^2+x^2) - \left(\frac{y^2+x^2}{y}\right)^{\cdot}\sqrt{(hx)} = 0, \text{ and put}$$

$x = \frac{y\sqrt{(cv-c^2)}}{c}$; then $\left(\frac{x}{y}\right)^{\cdot} = \frac{\dot{v}}{2\sqrt{(cv-c^2)}}$, $x^2+y^2 = \frac{v}{c}y^2$, $\left(\frac{y^2+x^2}{y}\right)^{\cdot} = \left(\frac{vy}{c}\right)^{\cdot}$, $\sqrt{(hx)} = \sqrt{\left(\frac{hy}{c}\sqrt{(cv-c^2)}\right)}$, and by substitution $\frac{\dot{v}}{2\sqrt{(cv-c^2)}} \times \frac{v}{c}y^2 - \left(\frac{vy}{c}\right)^{\cdot} \times \sqrt{\left(\frac{hy}{c}\sqrt{(cv-c^2)}\right)} = 0$,

or $\frac{v\dot{v}}{2(cv-c^2)^{\frac{3}{4}}} = \frac{y\dot{v}+v\dot{y}}{y^{\frac{1}{2}}}$, and dividing by $v^{\frac{3}{2}}$,

$$\frac{v^{-\frac{1}{2}}\dot{v}}{2(cv-c^2)^{\frac{3}{4}}} = \frac{y\dot{v}+v\dot{y}}{y^{\frac{1}{2}}v^{\frac{3}{2}}}.$$

The fluent of $\frac{v^{-\frac{1}{2}}\dot{v}}{2(cv - c^2)^{\frac{3}{4}}}$ is assignable by arches of the conic sections, and the fluent of $\frac{y\dot{v} + v\dot{y}}{y^{\frac{3}{2}}v^{\frac{3}{2}}}$ is $= -2y^{-\frac{1}{2}}v^{-\frac{1}{2}}$; therefore if the fluent of the first expression be denoted by $V^{-\frac{1}{2}}$, we shall have

$$V^{-\frac{1}{2}} = -2y^{-\frac{1}{2}}v^{-\frac{1}{2}},\ v = \frac{V}{4y},\ \text{and}\ x = \frac{y}{c}\sqrt{\left(\frac{cV}{4y} - c^2\right)},$$

the equation of the synchronous curve.

The calculation being made for V, according to the ordinary methods, the curve may easily be constructed and the values of x and y determined as required. The problem may be resolved in a similar manner when DE is a curve, given in kind and position, instead of a straight line.

Messrs. Harvey *and* Watts, *and the Rev. Mr.* Scurr *answered it.*

XX. PRIZE QUESTION 310, *by Mr.* W. WALLACE.

Find such integer values of x, y, z as shall render the three expressions $x^2 + axy + y^2$, $x^2 + a'.xz + z^2$, $y^2 + a''yz + z^2$ squares, a, a', a'' being given numbers.

FIRST SOLUTION, *by a* LADY.

Assume $x = an^2 + 2mn$, or $x = an^2 - 2mn$,

$$y = m^2 - n^2,$$
$$z = a''n^2 + 2mn.$$

Then, by substituting these values in the first and last of the given expressions, we find

$$x^2 + axy + y^2 = (m^2 \pm amn + n^2)^2 \ldots\ldots\ldots\text{(A)}$$
$$y^2 + a''yz + z^2 = (m^2 + a'mn + n^2)^2 \ldots\ldots\ldots\text{(B)}.$$

This transformation renders two of the given expressions squares, and it only remains to make the third expression $x^2 + a'xz + z^2$ a square.

Let us put

$$an \pm 2m = r, \quad a''n + 2m = s;\ \ldots\ldots\ldots\ldots(1)$$

Then $x = nr$, and $z = ns$, and

$$x^2 + a'xz + z^2 = n^2(r^2 + a'rs + s^2).$$

Assume now

$$r = p^2 - q^2, \quad s = a'q^2 + 2pq;\ \ldots\ldots\ldots\ldots(2)$$

and we have

$$r^2 + a'rs + s^2 = (p^2 + a'pq + q^2)^2;$$

and consequently

$$x^2 + a'xz + z^2 = n^2(p^2 + a'pq + q^2)^2 \ldots\ldots\ldots\text{(C)}.$$

These three formulæ, (A), (B), (C) exhibit transformed values of the given expressions, which are squares, as required by the question; and it now only remains that we find the values of the quantities m and n in terms of p and q.

By comparing equations (1) and (2), we have

$$an \pm 2m = p^2 - q^2,$$
$$a''n + 2m = a'q^2 + 2pq.$$

From these equations, which are of the first degree in respect of m and n, we easily find, by known methods,

$$m = \frac{1}{2(a'' + a)}\left\{\pm a''p^2 \mp 2apq \mp (aa' + a'')q^2\right\},$$

$$n = \frac{1}{2(a'' \mp a)}\left\{2(a' \pm 1)\ q^2 + 4pq \mp 2p^2\right\}.$$

But as the values of x, y and z, assumed at the beginning of this solution, are all of the second degree, and homogeneous in respect of m and n, it is easy to see that any quantity by which both m and n are multiplied may be rejected, and still the transformed expressions for the quantities in the question will be squares; therefore, rejecting the common denominator $2(a'' \mp a)$, we have more simply, as follows:

If we assume $x = an^2 + 2mn$, then

$$\left.\begin{aligned} m &= a''p^2 - 2apq - (a'a + a'')\,q^2, \\ n &= 2(a' + 1)\,q^2 + 4pq - 2p^2. \end{aligned}\right\} \ldots\ldots\ldots (3)$$

But if we assume $x = an^2 - 2mn$, then,

$$\left.\begin{aligned} m &= (aa' + a'')q^2 + 2apq - a''p^2, \\ n &= 2(a' - 1)q^2 + 4pq + 2p^2. \end{aligned}\right\} \ldots\ldots\ldots (4)$$

From these, and the other two assumed formulæ, viz.

$$y = m^2 - n^2, \qquad z = a''n^2 + 2mn,$$

the values of x, y and z may always be expressed in terms of the two indeterminate quantities p and q, which may have any numerical values whatever.

If we suppose that $a = a' = a''$, the formulæ (3) will not apply, because they would give x and z each $= 0$. In this particular case we must have recourse to formulæ (4) by which we find

$$\left.\begin{aligned} m &= a\left\{(a + 2)\,q^2 - (p - q)^2\right\} \\ n &= 2\left\{(a + 2)\,q^2 + (p + q)^2\right\} \end{aligned}\right\} \ldots\ldots\ldots (5)$$

and these values of m and n give satisfactory values of x, y and z.

Ex. 1. Let the expressions to be made squares be

$$x^2 - xy + y^2,\ x^2 - xz + z^2,\ y^2 - yz + z^2.$$

Here $a = a' = a'' = -1$, therefore, by the formulæ (5),

$$m = (p - q)^2 - q^2; \qquad n = 2(p + q)^2 - 6q^2.$$

Take $p = 3$, $q = 1$, then $m = 3$, $n = 26$, and $x = -n^2 - 2mn = -832$, $y = m^2 - n^2 = -667$, $z = -n^2 + 2mn = -520$ these values are negative, but it is evident that the conditions of the question will be equally satisfied if we change the sign and make

$$x = 832, \quad y = 667, \quad z = 520.$$

Accordingly we have

$$x^2 - xy + y^2 = 763^2,$$
$$x^2 - xz + z^2 = 728^2,$$
$$y^2 - yz + z^2 = 607^2.$$

Ex. 2. Let the expressions to be made squares be

$$x^2 + 3xy + y^2, \quad x^2 + 3xz + z^2, \quad y^2 + 3yz + z^2.$$

In this case, by the formulæ (5),

$$m = 3\{5q^2 - (p - q)^2\}, \quad n = 2\{q^2 + (p + q)^2\}.$$

Take $p = 4$, $q = 3$, then $m = 132$, $n = 116$; but these values of m and n have a common factor 4, therefore we have in smaller numbers, $m = 33$, $n = 29$: and

$$x = 609, \quad y = 248, \quad z = 4437.$$
$$x^2 + 3xy + y^2 = 941^2,$$
$$x^2 + 3xz + z^2 = 5307^2,$$
$$y^2 + 3yz + z^2 = 4801^2.$$

Ex. 3. Let the expressions to be made squares be

$$x^2 + xy + y^2, \quad x^2 + 3xz + z^2, \quad y^2 + 7yz + z^2.$$

In this case, $a = 1$, $a' = 3$, $a'' = 7$, therefore, by the formulæ (3),

$$m = 7p^2 - 2pq - 10q^2, \quad n = 8q^2 + 4pq - 2p^2.$$

Let $p = 2$, $q = 1$, then, $m = 14$, $n = 8$; but as these values have a common factor 2, we may take $m = 7$, $n = 4$; hence

$$x = n^2 + 2mn = 72, \; y = m^2 - n^2 = 33, \; z = 7n^2 + 2mn = 168,$$

and leaving out the common factor 3,

$$x = 24, \quad y = 11, \quad z = 56.$$
$$x^2 + xy + y^2 = 31^2,$$
$$x^2 + 3xz + z^2 = 88^2,$$
$$y^2 + 7yz + z^2 = 87^2.$$

The Lady who has so ingeniously answered this question will receive the prize medal, by applying to Mr. Glendinning, No. 25, Hatton Garden.

SECOND SOLUTION, *by Mr.* LOWRY.

By the question, the three formulæ $x^2 + axy + y^2$, $x^2 + a'xz + z^2$, and $y^2 + a''yz + z^2$ are to be squares; therefore if the first and second be divided by x^2, and the third by y^2, we shall have $1 + a\frac{y}{x} + \frac{y^2}{x^2}$, $1 + a'\frac{z}{x} + \frac{z^2}{x^2}$ and $1 + a''\frac{z}{y} + \frac{z^2}{y^2}$, to make into squares. Assume $1 \mp \frac{vy}{ux}$ for the root of the first, and $1 \mp \frac{mz}{nx}$ for the root of the second; then, from the first assumption we get $\frac{y}{x} = \frac{u(au \pm 2v)}{v^2 - u^2}$, and from the second $\frac{z}{x} = \frac{n(a'n \pm 2m)}{m^2 - n^2}$; therefore by division $\frac{z}{y} = \frac{n(a'n \pm 2m)\,(v^2 - u^2)}{u(au \pm 2v)\,(m^2 - n^2)}$ and the third formula becomes

$$1 + a''\frac{n(a'n \pm 2m)\,(v^2 - u^2)}{u(au \pm 2v)\,(m^2 - n^2)} + \left(\frac{n(a'n \pm 2m)\,(v^2 - u^2)}{u(au \pm 2v)\,(m^2 - n^2)}\right)^2.$$

This expression is rather complex, but it may be simplified by making $v^2 - u^2 = m^2 - n^2$, which we are evidently permitted to do. Now the most obvious way to do this, is, to make $v = m$ and $u = n$; then will $\frac{z}{y}$ be $= \frac{a'n \pm 2m}{an \pm 2m}$, and the expression to be made a square $= 1 + a''\frac{a'n \pm 2m}{an \pm 2m} + \left(\frac{a'n \pm 2m}{an \pm 2m}\right)^2$. Assume the root $= \frac{r}{s} - \frac{a'n \pm 2m}{an \pm 2m}$, then we shall have $1 + a''\frac{a'n \pm 2m}{an \pm 2m} = \frac{r^2}{s^2} - \frac{2r}{s}\left(\frac{a'n \pm 2m}{an \pm 2m}\right)$, and by reduction $(a''s^2 + 2rs) \times (a'n \pm 2m) = (an \pm 2m)\,(r^2 - s^2)$, which equation will be satisfied by making $a'n \pm 2m = r^2 - s^2$, and $an \pm 2m = a''s^2 + 2rs$. Or, if $1 - \frac{e}{t}\left(\frac{a'n \pm 2m}{an \pm 2m}\right)$ be assumed for the root, we shall have $(a''t^2 + 2te)\,(an \pm 2m) = (a'n \pm 2m)(e^2 - t^2)$, which will be satisfied by making $a'n \pm 2m = a''t^2 + 2te$, and $an \pm 2m = e^2 - t^2$. Put $p = r^2 - s^2$ or $a''t^2 + 2te$, and $q = a''s^2 + 2rs$ or $e^2 - t^2$, r and s, or e and t, being any numbers whatever; then from the equations $a'n \pm 2m = p$ and $an \pm 2m = q$, we get, by elimination,

$$m = \frac{a'q \backsim ap}{2(a' - a)}, \text{ or } \frac{ap \backsim a'q}{2(a' + a)}, \text{ and } n = \frac{p \backsim q}{a' - a}, \text{ or } \frac{p + q}{a' + a}.$$

Now since $v = m$, and $u = n$, $\frac{y}{x}$ is $= n\left(\frac{an \pm 2m}{m^2 - n^2}\right)$, and $\frac{z}{x} = n\left(\frac{a'n \pm 2m}{m^2 - n^2}\right)$; we may therefore make $x = m^2 - n^2$, $y =$

$n(an \pm 2m)$, and $z = n\,(a'n \pm 2m)$: then substituting for m and n the above values, we get, after proper reduction,

$$\begin{aligned} x &= (a'^2 - 4)q^2 - (2aa' \mp 8)pq + (a^2 - 4)p^2, \\ y &= 4(a' \mp a)\,(pq \mp q^2), \qquad\qquad (1) \\ z &= 4(a' \mp a)\,(p^2 \mp pq). \end{aligned}$$

Since the quantities x, y and z are alike concerned in the expressions which are to be made squares, we may evidently make the following transformations, which will lead to other sets of answers.

Let $p = r^2 - s^2$ or $a't^2 + 2te$, and $q = a's^2 + 2rs$ or $e^2 - t^2$; then we have the equations $a''n \pm 2m = p$, and $an \pm 2m = q$; and $x = n\,(an \pm 2m)$, $y = m^2 - n^2$, $z = n\,(a''n \pm 2m)$. Whence

$$\begin{aligned} x &= 4(a'' \mp a)\,(pq \mp q^2), \\ y &= (a''^2 - 4)\,q^2 - (2aa'' \mp 8)\,pq + (a^2 - 4)p^2, \qquad (2) \\ z &= 4(a'' \mp a)\,(p^2 \mp pq). \end{aligned}$$

Also if $p = r^2 - s^2$ or $at^2 + 2te$, and $q = as^2 + 2rs$ or $e^2 - t^2$, we shall have $a'n \pm 2m = p$, and $a''n \pm 2m = q$; $x = n\,(a'n \pm 2m)$, $y = n\,(a''n \pm 2m)$, $z = m^2 - n^2$; whence

$$\begin{aligned} x &= 4(a' \mp a'')\,(p^2 \mp pq), \\ y &= 4(a' \mp a'')\,(pq \mp q^2), \qquad\qquad (3) \\ z &= (a'^2 - 4)q^2 - (2a''a' \mp 8)pq + (a''^2 - 4)p^2. \end{aligned}$$

Any of which sets of formulæ may be used that will bring out values of x, y and z that are all positive or all negative.

Ex. 1. Suppose $a = 2$, $a' = 1$, and $a'' = 3$; and take $r = 3$ and $s = 1$; then, by formula (1), $p = 8$, $q = 9$; $x = 45$, $y = 36$ and $z = 32$; and we have $x^2 + 2xy + y^2 = 77^2$, $x^2 + xz + z^2 = 67^2$ and $y^2 + 3yz + z^2 = 76^2$.

Ex. 2. Suppose $a = 4$, $a' = 5$ and $a'' = 6$; and take $r = 4$ and $s = 1$; then, by formula (1), $p = 15$, $q = 14$, $x = 96$, $y = 56$, and $z = 60$; or, dividing by the common factor 4, $x = 24$, $y = 14$ and $z = 15$; whence $x^2 + 4xy + y^2 = 46^2$, $x^2 + 5xz + z^2 = 51^2$, and $y^2 + 6yz + z^2 = 41^2$.

Ex. 3. Suppose $a = -2$, $a' = -1$, $a'' = -3$, and take $r = 2$ and $s = 1$; then, by formula (1), $p = 3$, $q = 1$, $x = 9$, $y = 8$ and $z = 24$: hence $x^2 - 2xy + y^2 = 1^2$, $x^2 - xz + z^2 = 21^2$, and $y^2 - 3yz + z^2 = 8^2$.

Or if we take $r = 3$ and $s = 1$; then, by formula (2), $p = 8$, $q = 5$, $x = -60$, $y = -35$, and $z = -96$; or, changing the sign, $x = 60$, $y = 35$, and $z = 96$, which are another set of numbers that will answer, for $x^2 - 2xy + y^2 = 25^2$, $x^2 - xz + z^2 = 84^2$, and $y^2 - 3yz + z^2 = 19^2$.

Again, by taking $r = 2$ and $s = 1$, we get, from formula (3), $p = 3$, and $q = 6$; or (dividing by the common factor 3), $p = 1$, and $q = 2$; whence $x = 8$, $y = 16$, and $z = 3$, which values will also answer. For $x^2 - 2xy + y^2 = 8^2$, $x^2 - xz + z^2 = 7^2$ and $y^2 - 3yz + z^2 = 11^2$.

Ex. 4. Let $a=1$, $a'=3$ and $a''=7$, and take $t=1$ and $e=2$; then, by formula (3), $p=7$, $q=3$, $x=12\times 24$, $y=12\times 11$, and $z=12\times 56$, or dividing by 12, $x=24$, $y=11$ and $z=56$; hence $x^2+xy+y^2=31^2$, $x^2+3xz+z^2=88^2$, and $y^2+7yz+z^2=87^2$.

Ex. 5. Suppose $a=a'=a''=-1$. In this case, where the given numbers are equal, any of the formulæ will answer, provided the lower or positive sign be used. Hence taking $r=3$ and $s=2$, we have, by formula (1), $p=5$ and $q=2rs-s^2=8$; $x=-832$, $y=-667$, and $z=-520$, which values may be changed from negative to positive; then $x^2-xy+y^2=763^2$, $x^2-xz+z^2=728^2$, and $y^2-yz+z^2=607^2$.

Ex. 6. Let $a=a'=a''=5$, and take $r=2$ and $s=1$; then, by formula (1), $p=3$, $q=9$, $x=6\times 9$, $y=36\times 120$, and $z=36\times 40$; therefore, omitting the common factor 36, we have $x=9$, $y=120$ and $z=40$. Hence $x^2+5xy+y^2=141^2$; $x^2+5xz+z^2=59^2$ and $y^2+5yz+z^2=200^2$.

This solution, it must be observed, is much restricted by supposing $v=m$ and $u=n$, and innumerable values of x, y and z may exist that will answer the conditions of the question, which are not included in the preceding formulæ. There are however several ways in which the question may be resolved more generally. If, instead of taking $v=m$, and $u=n$, we make $v^2-u^2=m^2-n^2$, by means of the arbitrary factors r and s, as in the solution to question 294, we shall have $v=\frac{r^2+s^2}{2rs}m+\frac{r^2-s^2}{2rs}n$, and $u=\frac{r^2-s^2}{2rs}m+\frac{r^2+s^2}{2rs}n$; or, $v=pm+qn$, and $u=qm+pn$, putting $p=\frac{r^2+s^2}{2rs}$ and $q=\frac{r^2-s^2}{2rs}$; then $u(au+2v)=(aq^2+2pq)m^2+2(p^2+q^2+apq)mn+(ap^2+2pq)n^2$. Now when v^2-u^2 is $=m^2-n^2$, $1+a''\frac{n(a'n+2m)}{u(au+2v)}+\frac{n^2}{u^2}\left(\frac{a'n+2m}{au+2v}\right)^2$, must be a square. Let its root be assumed $=1-\frac{e}{wt}\cdot\frac{n(a'n+2m)}{u(au+2v)}$; then, by reduction we get $u(au+2v)(a''w^2t^2+2wte)=n(a'n+2m)(e^2-w^2t^2)$, which equation will be satisfied by making $n(a'n+2m)=a''wt^2+2te$, and $wu(au+2v)=e^2-w^2t^2$. Hence we have the two equations $n(a'n+2m)=a''wt^2+2te$, and $w(aq^2+2pq)m^2+2w(p^2+q^2+apq)mn+w(ap^2+2pq)n^2=e^2-w^2t^2$; to determine the value of m and n. Now if we proceed to eliminate these quantities by the ordinary rules, we come to an equation of the fourth degree, but as this will not answer the purpose, we must assume such a relation between two of the indeterminate quantities as will enable us, if possible, to find m and n by a simple equation. If we take $t = n$, then will $a'n + 2m$ be $= a''nw + 2e$, or $e = \frac{a' - a''w}{2} n + m$; therefore

$$e^2 - w^2t^2 = \tfrac{1}{4}(a'^2 - 2a'a''w + (a''^2 - 4)w^2)n^2 + (a' - a''w)mn + m^2$$

$$= w(aq^2 + 2pq)m^2 + 2w(p^2 + q^2 + apq)mn + w(ap^2 + 2pq)n^2$$

and making $w(aq^2 + 2pq)m^2 = m^2$, in order that the equation may be divisible by n, we have $w = \frac{1}{aq^2 + 2pq}$, and by reduction

$$\left(\frac{a'^2}{4w^2} - \frac{2a'a''}{4w} + \frac{a''^2 - 4}{4} - \frac{ap^2 + 2pq}{w}\right)n = \left(\frac{2p^2 + 2q^2 + 2apq}{w} - \frac{a'}{w} + a''\right)m.$$

Therefore to satisfy this equation we may take

$$m = \tfrac{1}{4}\left(\frac{a'^2}{w^2} - \frac{2a'a''}{w} + a''^2 - 4\right) - \frac{ap^2 + 2pq}{w},$$

$$n = \frac{2}{w}\left(p^2 + q^2 + apq\right) - \frac{a'}{w} + a'', \text{ where } p = \frac{r^2 + s^2}{2rs},\ q = \frac{r^2 - s^2}{2rs}$$

and $w = \frac{1}{aq^2 + 2pq} = \frac{4r^2s^2}{a(r^2 - s^2)^2 + 2(r^4 - s^4)}$, r and s being any numbers whatever. Then the general values of x, y and z will be as follow, viz.

$$x = m^2 - n^2,$$

$$y = \frac{1}{4}\left(\frac{a'^2}{w} - 2a'a'' + (a''^2 - 4)w\right)n^2 + \left(\frac{a'}{w} - a''\right)mn + \frac{m^2}{w},$$

$$z = n(a'n + 2m).$$

When the second terms are wanting, or the co-efficients a, a', a' are each $= 0$, we have $w = \frac{1}{2pq}$, $m = -(1 + 4p^2q^2)$ and $n = 4qp(p^2 + q^2)$; or, substituting for p and q their proper values and omitting the factors that are common to each expression, $m = -(r^4 + s^4)$, $n = 2(r^4 - s^4)$ and $w = \frac{4r^2s^2}{n}$; whence

$$x = 2r^2s^2(m^2 - n^2),\ y = \frac{n}{2}(m^2 - 16r^4s^4) \text{ and } z = 4mnr^2s^2.$$

For example, if $r^2 = 2$ and $s = 1$, then $m = -5$, $n = 6$; $x = 4(25 - 36) = -41$; $y = 3(25 - 64) = -117$; and $z = 4 \times -30 \times 2 = -240$: hence, changing the sign, $x = 44$, $y = 117$ and $z = 240$; $x^2 + y^2 = 125^2$, $x^2 + z^2 = 244^2$ and $y^2 + z^2 = 267^2$.

The question may also be resolved very generally as follows :

If in the general expression $\frac{z}{y}$ we put $r = n(a'n + 2m)$ and $s = m^2 - n^2$, we shall have $1 + a'' \frac{r(v^2 - u^2)}{su(au + 2v)} + \frac{r^2(v^2 - u^2)^2}{s^2u^2(au + 2v)^2}$ to make into a square: therefore multiplying by $s^2u^2(au + 2v)^2$, $s^2u^2(au + 2v)^2 + a''rsu(au + 2v)(v^2 - u^2) + r^2(v^2 - u^2)^2$ must be a square. By making the necessary multiplications, and arranging the terms according to the powers of v, this expression becomes $r^2v^4 + 2a''rsuv^3 + (4s^2 + aa''rs - 2r^2)u^2v^2 + (4s^2a - 2a''rs)u^3v + (r^2 - a'a'rs + a^2s^2)u^4$, or $r^2v^4 + 2a''rsuv^3 + \alpha u^2v^2 + \beta u^3v + \delta u^4$, putting for the present $\alpha = 4s^2 + aa''rs - 2r^2$, $\beta = 4s^2a - 2a''rs$, and $\delta = r^2 - aa''rs + a^2s^2$.

Now to make this expression a square, let it be assumed $= (rv^2 + pv + q)^2 = r^2v^4 + 2rpv^3 + (p^2 + 2rq)v^2 + 2pqv + q^2$; then, by comparing the co-efficients of the homologous terms, in order that the three first terms may destroy one another, we have $2a''rsu = 2rp$, $\alpha u^2 = p^2 + 2rq$, $\beta u^3v + \delta u^4 = 2pqv + q^2$: whence $p = a''su$, $q = \frac{\alpha u^2 - p^2}{2r} = (\alpha - a''^2s^2)\frac{u^2}{2r}$, and $v = \frac{q^2 - \delta u^4}{\beta u^3 - 2pq} = \frac{((\alpha - a''^2s^2)^2 - 4r^2\delta)u}{4r^2\beta - 4ra''s(\alpha - a''^2s^2)}$.

Therefore we may take $v = (\alpha - a''^2s^2)^2 - 4r^2\delta$, and $u = 4r^2\beta - 4ra''s(\alpha - a''^2s^2)$, and if we substitute for α, β and δ their values in terms of r and s, we shall have, after proper reduction,

$$v = (4 - a''^2)s^2 + 2aa''rs - (4 + a^2)r^2,$$
$$u = 4r(ar - a's);$$

and the general values of x, y and z will then be as follow, viz.

$$x = s(v^2 - u^2),$$
$$y = su(au + 2v),$$
$$z = r(v^2 - u^2);$$

where $r = n(an' + 2m)$, and $s = m^2 - n^2$: m and n being any indeterminate numbers taken at pleasure.

In the particular case of $a = a' = a'' = -1$, besides the answers contained in the above formulæ, we may deduce other sets. For r is then $= n(2m - n)$, $\alpha = 4s^2 + rs - 2r^2$, $\beta = 2rs - 4s^2$, and $\delta = r^2 - rs + s^2 =$, in this case, to $(n^2 - nm + m^2)^2$; hence if we put $d = n^2 - nm + m^2$, the expression to be made a square will be $r^2v^4 - 2rsuv^3 + \alpha u^2v^2 + \beta u^3v + d^2u^4$, where both the first and last terms are squares. We may therefore assume the root $= rv^2 - pvu \pm du^2$, then the square is $= r^2v^4 - 2rpuv^3 + (p^2 \pm 2rd)u^2v^2 \mp 2pdu^3v + d^2u^4$; and making $\beta u^3 = \mp 2pdu^3$, we have $-2rsuv^3 + \alpha u^2v^2 = -2rpuv^3 + (p^2 \pm 2rd)u^2v^2$.

therefore $p = \frac{\mp\beta}{2d}$, and $v = \frac{(\alpha - (p^2 \pm 2rd))u}{2r(s-p)} = \frac{(2 \mp 2rd)4d^2 - \beta^2)u}{4rd(2 \cdot d \pm \beta)}$. Hence we may take $v = 4d^2(\alpha \mp 2rd) - \beta^2$, and $u = 4rd(2sd \pm \beta)$; then, substituting for α and β their proper values, we have

$$v = d^2(4s^2 + rs - 2r^2 \mp 2rd) - (r + 2s)^2 s^2,$$
$$u = 2rsd^2 \pm rd(2rs - 4s^2).$$

Again, instead of equating the co-efficients of the first power of v, we may equate those of the third; then we shall have,

$$2rs = 2rp, \text{ or } s = p,$$

and $\alpha u^2 v^2 + \beta u^3 v = (p^2 \pm 2rd)u^2 v^2 \mp 2pdu^3 v$;

whence $v = \frac{(\beta \pm 2pd)u}{p^2 \pm 2rd - \alpha} = \frac{(\beta \pm 2\ d)u}{s^2 \pm 2rd - \alpha}$; therefore we may take $v = \beta \pm 2 \cdot d$, and $u = s^2 \pm 2rd - \alpha$; then, by substitution,

$$v = 2s(r \pm d - 2s),$$
$$u = r(2r - s \pm 2d) - 3s^2.$$

Ex. Take $m = 3$ and $n = 2$, then $r = n(2m - n) = 8$, $s = m^2 - n^2 = 5$, $d = n^2 - nm + m^2 = 7$, $v = 2s(r + d - 2s) = 50 = 25 \times 2$, $u = r(2r - s + 2d) - 3s^2 = 125 = 25 \times 5$; hence, omitting the common factor 25, we have $v = 2$, and $u = 5$, $x = s(v^2 - u^2) = -105$, $y = su(2v - u) = -25$, and $z = r(v^2 - u^2) = -168$; therefore changing the sign, $x = 105$, $y = 25$ and $z = 168$, and $x^2 - xy + y^2 = 95^2$, $x^2 - xz + z^2 = 147^2$ and $y^2 - yz + z^2 = 157^2$.

The problem may be resolved in a similar manner, when an arbitrary co-efficient is prefixed to the last term of each of the formulæ in the question, or when it is required that $x^2 + axy + by^2$, $x^2 + a'xz + b'z^2$ and $y^2 + a''yz + b''z^2$ shall be squares. For making the same substitution as before, we have $\frac{y}{x} = \frac{u(au \pm 2v)}{v^2 - bu^2}$, $\frac{z}{x} = \frac{n(a'n \pm 2m)}{m^2 - b'n^2}$ and $\frac{z}{y} = \frac{n(a'n \pm 2m)(v^2 - bu^2)}{u(au \pm 2v)(m^2 - b'n^2)}$. But since we cannot in general make $v^2 - bu^2 = m^2 - b'n^2$, at least in a way that will answer our purpose, we shall begin with the particular case when $b = b'$; and here if we make $v = m$ and $u = n$, $v^2 - bu^2$ will be $= m^2 - bn^2$, $\frac{z}{y} = \frac{a'n \pm 2m}{an \pm 2m}$, and we shall have to make $1 + a'' \frac{a'n \pm 2m}{an \pm 2m} + b'' \left(\frac{a'n \pm 2m}{an \pm 2m}\right)^2$ into a square. Let its root be assumed $= 1 - \frac{e}{t}\left(\frac{a'n \pm 2m}{an \pm 2m}\right)$, then, we have, as before, $a'n \pm 2m = a''t^2 + 2te$, and $an \pm 2m = e^2 - b''t^2$; whence $m = \frac{a'q \backsim ap}{2(a' \mp a)}$, $n = \frac{p \mp q}{a' \mp a}$, $x = m^2 - bn^2$, $y = n(an \pm 2m)$, and $z = n(a'n \pm 2m)$.

Or,

$$\begin{aligned} x &= (a^2 - 4b)q^2 - (2aa' \mp 8b)pq + (a^2 - 4b)p^2, \\ y &= 4(a' \mp a)(pq \mp q^2), \\ z &= 4(a' \mp a)(p^2 \mp pq). \end{aligned} \tag{4}$$

This formula serves to resolve the problem when $x^2 + axy + by^2$, $x^2 + axz + bz^2$ and $y^2 + a''yz + b''z^2$ are required to be squares.

And making similar transformations to those which were made in the first part of the solution, we shall obtain formulæ that will answer in other cases. Thus let $p = a't^2 + 2te$, and $q = e^2 - b't^2$, then $a''n \pm 2m = p$, $an \pm 2m = q$, $x = n(an \pm 2m)$, $y = m^2 - b'n^2$, $z = n(a''n \pm 2m)$: whence

$$\begin{aligned} x &= 4(a'' \mp a)(pq \mp q^2), \\ y &= (a''^2 - 4b)q^2 - (2aa'' \mp 8b)pq + (a^2 - 4b)p^2, \\ z &= 4(a'' \mp a)(p^2 \mp pq). \end{aligned} \tag{5}$$

This formula serves to resolve the problem when $x^2 + axy + by^2$, $x^2 + a'xz + b'z^2$ and $y^2 + a'yz + bz^2$ are required to be squares.

Again, put $p = at^2 + 2te$, and $q = e^2 - bt^2$; then $a'n \pm 2m = p$, $a'n \pm 2m = q$; $x = n(a'n \pm 2m)$, $y = n(a'n \pm 2m)$, $z = m^2 - b'n^2$; whence

$$\begin{aligned} x &= 4(a' \mp a'')(p^2 \mp pq), \\ y &= 4(a' \mp a'')(pq \mp q^2), \\ z &= (a'^2 - 4b')q^2 - (2a''a' \mp 8b)pq + (a''^2 - 4b)p^2. \end{aligned} \tag{6}$$

This formula may be used when $x^2 + axy + by^2$, $x^2 + a'xz + b'z^2$ and $y^2 + a''yz + b'z^2$ are required to be squares.

In the particular case when $b = b' = b''$, any of the formulæ may be made use of, observing, as before, that the positive sign must always be used when the co-efficients of the second terms are equal to each other.

Ex. 1. Suppose the co-efficients of the second terms to be each $= 1$, and those of the third $= -1$; then taking $t = 2$ and $e = 1$, we have $p = t^2 + 2et = 8$ and $q = e^2 + t^2 = 5$; whence $x = 685$, $y = 520$, $z = 832$; $x^2 + xy - y^2 = 745^2$, $x^2 + xz - z^2 = 589^2$, and $y^2 + yz - z^2 = 104^2$.

Ex. 2. Suppose the co-efficients of the second terms to be each $= 2$, and those of the third $= -1$; then taking $t = 2$ and $e = -1$, we have $p = 2t^2 + 2et = 4$, and $q = e^2 + t^2 = 5$: whence $x = 41$, $y = 90$ and $z = 72$, $x^2 + 2xy - y^2 = 31^2$, $x^2 + 2xz - z^2 = 126^2$, and $y^2 + 2yz - z^2 = 49^2$.

Ex. 3. Suppose the third terms are wanting, and that the co-efficients of the second terms are each $= 1$; then taking $t = 2$ and $e = 1$, we have $p = 8$ and $q = 1$; whence $x = (p - q)^2 = 49$, $y = 72$ and $z = 576$, $x^2 + xy = 77^2$, $x^2 + xz = 175^2$, and $y^2 + yz = 216^2$.

In making $v^2 - bu^2 = m^2 - b'n^2$, if we take $v = m$, then $b\,u$ is $= b'n^2$, or $u = n\sqrt{\frac{b'}{b}}$; and this suggests another case of the problem to which the preceding formulæ are applicable, viz. when $\frac{b'}{b}$ is a square, or $= d^2$, for then $u = dn$, $p = a'n \pm 2m$, $q = d\,(an \pm 2m)$ and the values of x, y, and z will be the same as before. But to proceed more generally in the case when two of the co-efficients are equal, or $b = b'$, if we make $v^2 - bu^2 = m^2 - bn^2$, then $v^2 - m^2 = b(u^2 - n^2)$, and by resolving these expressions into their factors, and introducing the arbitrary quantities r and s, we have $r\,(v + m) = sb\,(u + n)$ and $s(v - m) = r(u - n)$: whence $v = \frac{(r^2 + bs^2)\,n - 2rsm}{r^2 - bs^2}$, and $u = \frac{2rsn - (r^2 + bs^2)m}{r^2 - bs^2}$; or putting $q = \frac{r^2 + bs^2}{r^2 - bs^2}$ and $p = \frac{2rs}{r^2 - bs^2}$, $v = qn - pm$ and $u = pn - qm$: therefore $(au + 2v) = (ap^2 + 2bpq)n^2 - 2(apq + p^2 + bq^2)mn + (aqw^2 + 2pq)m^2$; and we have as before $n(a'n + 2m) = a''wt^2 + 2te$, and $w(ap^2 + 2bpq)n^2 - 2w(apq + p^2 + bq^2)mn + w(aq^2 + 2pq)m^2 = e^2 - b''w^2t^2$; therefore taking $t = n$, we have $e = \frac{a' - a''w}{2}n + m$ and $e^2 - b''w^2t^2 = \frac{1}{4}(a'^2 - 2a'a''w + (a''^2 - 4b)\,w^2)\,n^2 + (a' - a''w)mn + m^2 = w(ap^2 + 2bpq)n^2 - 2w(apq + p^2 + bq^2)mn + w(aq^2 + 2pq)m^2$. Hence making $w(aq^2 + 2pq)\,m^2 = m^2$, we have $w = \frac{1}{aq^2 + 2pq}$, and, reducing the equation as before,

$$m = \tfrac{1}{4}\left(\frac{a'^2}{w^2} - \frac{2a'a''}{w} + a''^2 - 4b''\right) > \frac{ap^2 + 2bpq}{w},$$

$$n = \frac{2(apq + p^2 + bq^2)}{w} + \frac{a'}{w} - a'';$$

then $x = m^2 - bn^2$, $z = n(a'n + 2m)$, and

$$y = \tfrac{1}{4}\left(\frac{a'^2}{w} - 2a'a'' + (a''^2 - 4b)w\right)n^2 + \left(\frac{a'}{w} - a''\right)mn + \frac{m^2}{w},$$

or $(ap^2 + 2bpq)n^2 - 2(apq + p^2 + bq^2)mn + (aq^2 + 2pq)\,m^2$.

When the second terms are wanting, or $a = a' = a'' = 0$, w is $= \frac{1}{2pq}$, $m = -(b'' + 4bp^2q^2)$, $n = 4pq(p^2 + bq^2)$, $x = m^2 - bn^2$, $y = 2pqm^2 - \frac{bn^2}{2pq}$, and $z = 2mn$, or multiplying by $2pq$, $x = 2pq(m^2 - bn^2)$, $y = 4p^2q^2m^2 - bn^2$, and $z = 4mnpq$.

These formulæ will not however, in all cases, lead to the smallest numbers, on account of the restricted values of p and q; but, from what has been done above, we may easily derive others that will give smaller values.

For supposing $x = m^2 - bn^2$ and $z = 2mn$, as before, if we put $x^2 + by^2 = c^2$ and $x^2 + bz^2 = d^2 = (m^2 + bn^2)$; then $x^2 = c^2 - by^2 = d^2 - bz^2$, or $c^2 - d^2 = b(y^2 - z^2)$; whence making $r(c + d) = sb(y + z)$ and $s(c - d) = r(y - z)$, $y = \frac{(r^2 + bs^2)z - 2rsd}{r^2 - bs^2} = qz - pd = 2qmn - p(m^2 + bn^2)$; therefore to make $1 + b''\frac{z^2}{y^2}$ into a square, we have, as before, $2mn = 2te$, $w(2qmn) - wp(m^2 + bn^2) = e^2 - b''w^2t^2 = m^2 - b''w^2n^2$, putting $n = t$ and $e = m$; and if we make $-wpm^2 = m^2$, then $w = -\frac{1}{p}$, $\frac{2q}{p}m = \left(b + \frac{b''}{p^2}\right)n$, or $2pqm = (bp^2 + b')n$; therefore, writing for p and q their proper values, we may take

$$m = bp^2 + b'' = 4br^2s^2 + b''(r^2 - bs^2)^2,$$
$$n = 2pq = 4rs(r^2 + bs^2).$$

Then $x = (m^2 - bn^2)(r^2 - bs^2)$, $y = 2mn(r^2 + bs^2) - 2rs(m^2 + bn^2)$, and $z = 2mn(r^2 - bs^2)$.

Ex. 1. Suppose $b = b'' = 2$, then $m = 8r^2s^2 + 2(r^2 - 2s^2)^2 = 2(r^4 + 4s^4)$, and $n = 2rs(r^2 + 2s^2)$, and taking $r = 2$, $s = 1$, we have $m = 40$ and $n = 48$, or dividing by 8, $m = 5$ and $n = 6$; then $x = (m^2 - 2n^2)(r^2 - 2s^2) = 47 \times 2$, $y = 2mn(r^2 + 2s^2) - 2rs \times (m^2 + 2n^2) = 14 \times 2$, $z = 2mn(r^2 - 2s^2) = 60 \times 2$; hence we may take $x = 47$, $y = 14$ and $z = 60$: Then $x^2 + 2y^2 = 51^2$, $x^2 + 2z^2 = 97^2$ and $y^2 + 2z^2 = 86^2$.

Ex. 2. Let $b = b'' = -2$, then $m = -2(r^4 + 8r^2s^2 + 4s^4)$ and $n = 4rs(r^2 - 2s^2)$; and taking $r = 2$ and $s = 1$, we have $x = (m^2 + 2n^2)(r^2 + 2s^2) = 177 \times 6$, $y = 2mn(r^2 - 2s^2) - 2rs(m^2 - 2n^2) = 748$, $z = 2mn(r^2 + 2s^2) = 52 \times 6$: Hence, dividing by 2, $x = 531$, $y = 374$, and $z = 156$; $x^2 - 2y^2 = 47^2$, $x^2 - 2z^2 = 483^2$, and $y^2 - 2z^2 = 302^2$.

Ex. 3. Suppose the co-efficients of the last terms are each $= 1$, then, in this case, $m = 4r^2s^2 + (r^2 - s^2)^2 = (r^2 + s^2)^2$ and $n = 4rs(r^2 + s^2)$; therefore, dividing by the common factor $r^2 + s^2$, m may be taken $= r^2 + s^2$ and $n = 4rs$; then

$$x = (r^2 - s^2)(r^4 - 14r^2s^2 + s^4),$$
$$y = 2rs(3r^4 - 10r^2s^2 + 3s^4),$$
$$z = 8rs(r^4 - s^4).$$

And these are general values of x, y and z that will make $x^2 + y^2$, $x^2 + z^2$ and $y^2 + z^2$ into squares. If $r = 2$ and $s = 1$, then $x = 3(16 - 56 + 1) = 3 \times 39 = 117$, $y = 4(48 - 40 + 3) = 4 \times 11 = 44$ and $z = 16(16 - 1) = 16 \times 15 = 240$, the same numbers that were found before, and which I believe are the least that will answer the question. Their discovery has been attributed to Euler, but very improperly, for they are given in Dr. Saunderson's Algebra, which was written several years before Euler's. Saunderson died in 1739, and Euler's Algebra was not composed till after the year 1766, (See Dr. Hutton's Dictionary, article Euler). Saunderson's solution of the question is indeed a very ingenious one, and the result remarkably simple; it is this: If c denote the hypothenuse and a and b the legs of a rational right-angled triangle, then the numbers will be expressed by $4abc$, $a \times (4b^2 - c^2)$ and $b \times (4a^2 - c^2)$. For example, the numbers 3, 4 and 5 constitute a right-angled triangle, hence taking $a = 3$, $b = 4$ and $c = 5$,

$$4abc = 240,\ a \times (4b^2 - c^2) = 117 \text{ and } b \times (4a^2 - c^2) = 44.$$

Ex. 4. Suppose the co-efficients of the last terms are each $= -1$, then $m = -(r^4 + 6r^2s^2 + s^4)$, $n = 4rs(r^2 - s^2)$; $x = (m^2 + n^2)(r^2 + s^2)$, $y = 2mn(r^2 - s^2) - 2rs(m^2 - n^2)$, $z = 2mn(r^2 + s^2)$, and taking $r = 3$ and $s = 1$, we have $m = -136$, and $n = 96$, or dividing by 8, $m = -17$ and $n = 12$: whence $x = 2165$, $y = 2067$ and $z = 2040$; $x^2 - y^2 = 644^2$, $x^2 - z^2 = 725^2$, and $y^2 - z^2 = 333^2$. And from this answer we may immediately deduce another, for we may evidently take $y = 725$ and $z = 644$, x remaining as before, then $x^2 - y^2 = 2040^2$, $x^2 - z^2 = 2067^2$, $y^2 - z^2 = 333^2$.

But instead of dwelling on these particular cases, let us try to resolve the problem generally, when the co-efficients of the last terms are perfectly arbitrary; for this purpose put $r = n(a'n + 2m)$ and $s = m^2 - b'n^2$; then we have

$$1 + \frac{a''r(v^2 - bu^2)}{su(au + 2v)} + \frac{b''r^2}{s^2u^2}\left(\frac{v^2 - bu^2}{au + 2v}\right)^2 \text{ to make into a square;}$$

therefore $s^2u^2(au + 2v)^2 + a''rs(au^2 + 2vu)(v^2 - bu^2) + b''r^2(v^2 - bu^2)^2$, or, $b''r^2v^4+2a''rsuv^3+(4s^2+aa''rs-2b''r^2)u^2v^2+(4s^2a-2a''brs)u^3v$ $+(b''r^2-aa''brs+a^2s^2)u^4$ must be a square. Now we cannot make this expression a square unless the first or last term be a square, therefore either b'' or $b''r^2-aa''brs+a^2s^2$ must be a square. Now in the last expression r and s are not arbitrary quantities themselves, but functions of the arbitrary quantities m and n, we must therefore substitute these functions in place of r and s, and then determine the relation that must subsist between m and n, so as to make $b''r^2-aa''brs+a^2s^2$ a square. Putting $n(a'n+2m)$ for r, and $m^2-b'n^2$ for s in this expression, it becomes $a^2m^4-2aa''bnm^3+(4b''-2b'a^2-aa'a''b)n^2m^2$ $+(4a'b''+2aa''b)n^3m+(b''a'^2+aa'a''b-b^2a^2)n^4$, or a^2m^4 $-2aa''bnm^3+\alpha n^2m^2+\beta n^3m+\delta n^4$; therefore assuming am^2 $-pm+b$ for the root, and proceeding as before, we get

$$m=\frac{((\alpha-a''^2b^2)^2-4a^2\delta)n}{4a\beta+4aa''b(\alpha-a''^2b^2)},$$ and substituting for α, β and δ their proper values,

$$\frac{m}{n}=\frac{(4b''-2b'a^2-aa'a''b-a''^2b^2)^2-4a^2(b''a'^2+aa'a''b-b'^2a^2)}{4a(4aa'b''+2a^2a''b+4a''b''b-2a^2a''b'b-aa'a''b^2-a''^3b^3)}.$$

Therefore we may take for m and n any two numbers having the ratio of the numerator of this fraction to the denominator; then $b''r^2-aa'brs+a^2s^2$ will be a square, and its root am^2 $-pm+q=am^2-a''bmn+(4b''-2b'a^2-aa'a''b-a''^2b^2)\times$ $\frac{n^2}{2a}=d$. It remains now only to make $b''r^2v^4+2a'rsuv^3+$ $(4s^2+aa''rs-2b''r^2)u^2v^2+(4r^2a-2a'brs)u^3v+d^2u^4$, or $b''r^2v^4+$ $2a''rsuv^3+\alpha u^2v^2+\beta u^3v+d^2u^4$, into a square. Let the root be assumed $=pv^2+qvu+du^2$, then the square is $=p^2v^4+$ $2pqv^3u+(q^2+2pd)v^2u^2+2qdvu^3+d^2u^4$, and by comparing the homologous co-efficients, we have $q=\frac{\beta}{2d}$, $p=\frac{\alpha-q^2}{2d}$,

$$v=\frac{(a''rs-pq)u}{p^2-b''r^2}=\frac{(16a''rsd^4-4\beta d^2(4d^2\alpha-\beta^2)^2)u}{(4d^2\alpha-\beta^2)^2-64b''rd^6};$$ whence we may take $v=16a''rsd^4-4\beta d^2(4d^2\alpha-\beta^2)^2$ and $u=(4d^2\alpha-\beta^2)^2$ $-64b''rd^6$, α being $=4s^2+aa''rs-2b''r^2$ and $\beta=4sa-$ $2a''brs$; then the general values of x, y and z will be as follow, viz.

$$x=(m^2-b'n^2)(v^2-b'u^2),$$
$$y=(m^2-b'n^2)(au^2+2vu),$$
$$z=(a'n^2+2mn)(v^2-bu^2).$$

Additional Solutions to two of the preceding Questions, which were received too late to be inserted in their proper places.

VIII. QUESTION 298. THIRD SOLUTION, *by Mr.* R. J. DISHNEAGH,
Trinity College, Cambridge.

Let AT (fig. 105, pl. 3.) $=$ the given distance from the origin of the the co-ordinates of the given curves $= p$ $\therefore$ NT $= \frac{y\dot{x}}{\dot{y}}$ $\therefore$ AT $=$ NT $-$ AN or $\frac{y\dot{x}}{\dot{y}} - x = p$, in the given curves. Now if $y = f(x, a)$ be the general equation to the given curves, x and y denoting the corresponding rectangular co-ordinates, and a being constant for each curve but variable from one to another. From this equation find a value of $\frac{y\dot{x}}{\dot{y}} - x$ in terms of x and a and eliminate a from this result by means of the equation $y = f(x, a)$. Put the result $= p$: this evidently gives an equation expressing the relation of x and y in the new curve, involving the constant quantity p.

EXAMPLE 1. Suppose the curves are ellipses whose major and minor axes are both variable but preserve a constant ratio.

Case 1. When the tangent cuts the axis on the same side of the centre as A, as in fig. 106, pl. 3.

The equation to the ellipse is $y^2 = \frac{b^2}{a^2}(2ax - x^2)$ and let

$$a : b :: m : 1 \therefore b = \frac{a}{m} \therefore y^2 = \frac{1}{m^2}(2ax - x^2) \therefore y\dot{y} = \frac{1}{m^2}(a - x)\dot{x}$$

$$\therefore \frac{y\dot{x}}{\dot{y}} = \frac{m^2y^2}{a - x} = \frac{2ax - x^2}{a - x} \therefore \frac{y\dot{x}}{\dot{y}} - x = \frac{2ax - x^2}{a - x} - x = \frac{ax}{a - x}.$$

Now $m^2y^2 = 2ax - x^2 \therefore a = \frac{m^2y^2 + x^2}{2x}$ $\therefore$ by substitution we have

$$p = \frac{ax}{a - x} = \left(\frac{m^2y^2 + x^2}{2x} \times x\right) \div \left(\frac{m^2y^2 + x^2}{2x} - x\right) = \frac{m^2y^2x + x^3}{m^2y^2 - x^2},$$

$$\therefore m^2y^2p - px^2 = m^2y^2x + x^3 \therefore m^2y^2(p - x) = x^2(p + x),$$

$$\therefore y = \pm \frac{x}{m}\sqrt{\frac{p + x}{p - x}}$$ which is the equation to the new curve.

Now when $x = 0$, $y = \pm 0$ $\therefore$ the curve passes through A: and if $x = p$, $y = \pm \infty$ $\therefore$ taking AT′ $=$ AT, and drawing LTL′ perpendicular to AT, it is an asymptote to the curve, and the curve extends in two similar infinite branches on each side of the axis.

now $\frac{r^2\dot\rho - \rho^2\dot r}{\rho^2 r - r^2\rho} = \frac{r^2\dot\rho - \rho^2\dot r}{\rho r(\rho - r)} = \frac{r\dot\rho}{\rho(\rho - r)} - \frac{\rho\dot r}{r(\rho - r)} = \frac{\dot\rho}{\rho - r} - \frac{\dot\rho}{\rho} - \frac{\dot r}{\rho - r} - \frac{\dot r}{r}$

$= \frac{\dot\rho - \dot r}{\rho - r} - \frac{\dot\rho}{\rho} - \frac{\dot r}{r}$; whence taking fluents,

$$\text{l.}\,(\rho - r) - \text{l.}\,\rho - \text{l.}\,r = \text{l.}\,(\sin\varphi) - \text{l.}\,\text{c}.$$

$$\text{l.}\,\frac{\rho - r}{\rho r} = \text{l.}\,\frac{\sin\varphi}{\text{c}},$$

and passing from logarithms to numbers,

$$\frac{\rho - r}{\rho r} = \frac{\sin\varphi}{\text{c}}.$$

Now $\rho \overset{\prime}{=} r + \underset{\bullet}{r}$, whence $\frac{\rho - r}{\rho r} = \frac{\underset{\bullet}{r}}{\rho r} = -\left(\frac{1}{r}\right)_{\bullet}$. So that passing on to integrals we get $-\frac{1}{r} \overset{\prime}{=} -\frac{\sin\varphi}{2\text{c}} + \frac{1}{\psi}$, where $\frac{1}{\psi}$ is any function of φ that does not change while φ increases to $\varphi + \varpi$: multiplying by $-2r\text{c}\psi$,

$$2\text{c}\psi = \psi r \sin\varphi - 2\text{c}r,$$

which is a general equation to all the conic sections (including the circle) upon the supposition of ψ being an ordinary correction; for putting y for its equal $r\sin\varphi$, and $\sqrt{(x^2+y^2)}$ for r, we have

$$2\text{c}\psi = \psi y - 2\text{c}\sqrt{(x^2 + y^2)},$$

transposing and squaring

$$4\text{c}^2\psi^2 - 4\text{c}\psi^2 y + \psi^2 y^2 = 4\text{c}^2 x^2 + 4\text{c}^2 y^2, \text{ that is}$$

$$x^2 + \frac{4\text{c}^2 - \psi^2}{4\text{c}^2}\,y^2 - \frac{\psi^2}{\text{c}}\,y - \psi^2 = 0,$$

which is to an ellipse if $2\text{c} > \psi$, and to a circle if c be infinitely greater than ψ, for the equation is then reduced to $x^2 + y^2 - \psi^2 = 0$. It is to an hyperbola if $2\text{c} < \psi$ and to a parabola if $2\text{c} = \psi$.

* * *

ARTICLE IV.

Solutions to Questions proposed in Number XI.

I. QUESTION 311, *by Mr.* JOHN HYNES, *Dublin.*

To divide a given square number n^2, into two such parts that the sum of their squares and the sum of their cubes may both be rational squares.

FIRST SOLUTION, *by a* LADY.

Let x be one part, then $n^2 - x$ will be the other; hence $(n^2 - x)^2 + x^2$ and $(n^2 - x)^3 + x^3$, or $n^4 - 2n^2x + 2x^2$ and $n^2(n^4 - 3n^2x + 3x^2)$ must be squares. But it is evident that the latter would be a square if $n^4 - 3n^2x + 3x^2$ was a square. Let $n^4 - 3n^2x + 3x^2 = n^4 - 2n^2px + p^2x^2$ and $n^4 - 2n^2x + 2x^2 = n^4 - 2n^2qx + q^2x^2$, then $-3n^2 + 3x = -2n^2p + p^2x$ and $-2n^2 + 2x = -2n^2q + q^2x$; from these equations we obtain $x = \frac{3n^2 - 2n^2p}{3 - p^2}$ and $x = \frac{2n^2 - 2n^2q}{2 - q^2}$, therefore putting these values of x equal to each other, and dividing by n^2, we have $\frac{3 - 2p}{3 - p^2} = \frac{2 - 2q}{2 - q^2}$. In order to determine p and q in rational numbers, let us assume $3 - 2p = 2 - 2q$, then $3 - p^2 = 2 - q^2$, from the former equation we find $p = \frac{1 + 2q}{2}$, which substituted in the latter equation gives $2 - q^2 = 3 - \frac{1 + 4q + 4q^2}{4}$; whence we have $q = \frac{3}{4}$ and this gives p, or $\frac{1 + 2q}{2} = \frac{5}{4}$. By substituting these values of p and q in the equations $x = \frac{n^2(3 - 2p)}{3 - p^2}$ or $x = \frac{n^2(2 - 2q)}{2 - q^2}$ we get $x = \frac{8n^2}{23}$, hence $n^2 - x = \frac{15n^2}{23}$. Thus it appears that the numbers sought are $\frac{8n^2}{23}$ and $\frac{15n^2}{23}$.

Let $n = 10$, then $x = \frac{800}{23}$ and $n^2 - x = \frac{1500}{23}$, hence

$(n^2 - x)^2 + x^2 = \frac{2890000}{529} = \left(\frac{1700}{23}\right)^2$ and $(n^2 - x)^3 + x^3 = \frac{3887000000}{12167} = \frac{169000000}{529} = \left(\frac{13000}{23}\right)^2$.

Let $n = 2$, then $x = \frac{32}{23}$ and $n^2 - x = \frac{60}{23}$, hence $(n^2 - x)^2 + x^2 = \left(\frac{68}{23}\right)^2$ and $(n^2 - x)^3 + x^3 = \left(\frac{104}{23}\right)^2$.

Let $n = \frac{1}{3}$, then $x = \frac{8}{207}$ and $n^2 - x = \frac{15}{207}$, hence $(n^2 - x)^2 + x^2 = \left(\frac{13}{207}\right)^2$, and $(n^2 - x)^3 + x^3 = \left(\frac{17}{207}\right)^2$.

SECOND SOLUTION, *by Mr.* CUNLIFFE, *R. M. College.*

Let $\frac{n^2x}{x+y}$ and $\frac{n^2y}{x+y}$ denote the two parts, then by the question $\frac{n^4x^2 + n^4y^2}{(x+y)^2} = \frac{n^4}{(x+y)^2} \times (x^2 + y^2) =$ a square; therefore $x^2 + y^2 =$ a square.

Again, by the question $\frac{n^6x^3 + n^6y^3}{(x+y)^3} = \frac{n^6(x+y)(x^2 - xy + y^2)}{(x+y)^3} = \frac{n^6}{(x+y)^2} \times (x^2 - xy + y^2) =$ a square; therefore $x^2 - xy + y^2 =$ a square.

Wherefore the question is reduced to the finding of such rational values of x and y as shall make $x^2 + y^2$ and $x^2 - xy + y^2$ both rational squares.

Put $x = r^2 - s^2$ and $y = 2rs$, then $x^2 + y^2 = (r^2 - s^2)^2 + 4r^2s^2 = (r^2 + s^2)^2$, which being a square there only remains to make $x^2 - xy + y^2 = (r^2 + s^2)^2 - 2rs(r^2 - s^2) = r^4 - 2r^3s + 2r^2s^2 + 2rs^3 + s^4$ a square: assume $r^2 - rs + s^2$ for its root, that is, put $r^4 - 2r^3s + 2r^2s^2 + 2rs^3 + s^4 = (r^2 - rs + s^2)^2 = r^4 - 2r^3s + 3r^2s^2 - 2rs^3 + s^4$: whence $r = 4s$, $x = r^2 - s^2 = 15s^2$, $y = 2rs = 8s^2$, $\frac{n^2x}{x+y} = \frac{15n^2}{23}$ and $\frac{n^2y}{x+y} = \frac{8n^2}{23}$ two parts that will answer.

Other numbers that will answer may be found as follows: put $r = 4s - v$, then $r^2 + s^2 = v^2 - 8sv + 17s^2$, $r^2 - s^2 = v^2 - 8sv + 15s^2$, $2rs = 2s(4s - v)$; whence the general expression $(r^2 + s^2)^2 - 2rs\,(r^2 - s^2)$, to be made a square, becomes $v^4 - 14sv^3 + 74s^2v^2 - 178s^3v + 169s^4$. Take $v^2 + \frac{89}{13}sv - 13s^2$ for its root, that is, put $v^4 - 14sv^3 + 74s^2v^2 - 178s^3v + 169s^4 = (v^2 + \frac{89}{13}sv - 13s^2)^2 = v^4 + \frac{178}{13}sv^3 + \frac{89^2}{169}s^2v^2 - 26s^2v^2 - 178s^3v + 169s^4$: whence $v = \frac{2993s}{1560}$, $r = 4s - v = \frac{3243s}{1560}$, $\frac{n^2x}{x+y} = \frac{8083449n^2}{18201609}$ and $\frac{n^2y}{x+y} = \frac{10118160n^2}{18201609}$, other two parts that will answer.

THIRD SOLUTION, *by Mr.* LOWRY, *R. M. College.*

Let x and y represent the parts required, then $x^2 + y^2$ and $x^3 + y^3$ must be squares, or since $x^3 + y^3 = (x + y)(x^2 + y^2 - xy) = n^2(x^2 + y^2 - xy)$, we have only to make $x^2 + y^2$ and $x^2 + y^2 - xy$ into squares. Now $x = n^2 - y$, and this value being substituted for x we have

$$x^2 + y^2 = n^4 - 2n^2y + 2y^2,$$

$$\text{and}\quad x^2 + y^2 - xy = n^4 - 3n^2y + 3y^2.$$

To make the first a square, assume the root $= n^2 - \frac{r}{s}y$, then $n^4 - 2n^2y + 2y^2 = n^4 - \frac{2r}{s}n^2y + \frac{r^2}{s^2}y^2$, and by reduction $(r^2 - 2s^2)y = 2(rs - s^2)n^2$; therefore, to make $n^4 - 2n^2y + y^2$ a square, we may take $n^2 = r^2 - 2s^2$, and $y = 2rs - 2s^2$.

Again, to make the second expression a square, assume the root $= n^2 - \frac{e}{tw}y$, then by a similar process $\left(\frac{e^2 - 3t^2w^2}{w}\right)y = (2et - 3t^2w)n^2$, therefore $n^4 - 3n^2y + 3y^2$ will be a square if $n^2 = \frac{e^2 - 3t^2w^2}{w}$ and $y = 2et - 3t^2w$. We have, therefore, by equating the values of n^2 and y,

$$w(r^2 - 2s^2) = e^2 - 3t^2w^2$$

$$\text{and}\quad 2rs - 2s^2 = 2et - 3t^2w,$$

to find the relation of r and s. To do this, suppose $s = t$, then the equations become

$$w(r^2 - 2s^2) = e^2 - 3s^2w^2$$

$$\text{and}\quad 2r - 2s = 2e - 3sw;$$

From the first, $e^2 = w(r^2 - 2s^2) + 3s^2w^2$,
and from the second, $4e^2 = 4r^2 + 4(3w - 2)rs + (3w - 2)^2s^2$;
therefore $4w(r^2 - 2s^2) + 12s^2w^2 = 4r^2 + 4(3w - 2)rs + (3w - 2)^2s^2$;
and making $4wr^2 = 4r^2$, in order that the equation may be divisible by s, we have $w = 1$; whence

$$-8s + 12s = 4r + s$$

and $\quad 4r = 3s$; therefore we may take $r = 3$ and $s = 4$; then $y = \frac{2(rs - s^2)m^2}{r^2 - 2s^2} = \frac{8}{23}n^2$ and $x = n^2 - y = \frac{15}{23}n^2$ and these parts will fulfil all the conditions of the question; for their sum is $= n^2$, the sum of their squares $= \left(\frac{17}{23}n^2\right)^2$ and the sum of their cubes $= \left(\frac{13n^2}{23}\right)^2 \times n^2$.

This question was also answered by Messrs. Adams, Eratosthenes, *and* Hynes.

II. QUESTION 312, *by Mr.* PHELAN, *Carlow*.

Required the least integer value of x that will make $103x^2 + 1$ a rational square?

FIRST SOLUTION, *by Mr.* CUNLIFFE.

$$\sqrt{103} = 10 + \cfrac{1}{6 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{20, \&c.}}}}}}}}}}}}$$

and the series of fractions converging to the value of $\sqrt{103}$, formed from the preceding continued fraction are

$$\overset{1}{\frac{10}{1}},\ \overset{1}{\frac{61}{6}},\ \overset{2}{\frac{71}{7}},\ \overset{1}{\frac{203}{20}},\ \overset{1}{\frac{274}{27}},\ \overset{9}{\frac{477}{47}},\ \overset{1}{\frac{4567}{450}},\ \overset{1}{\frac{5044}{497}},\ \overset{2}{\frac{9611}{947}},$$

$$\overset{1}{\frac{24266}{2391}},\ \overset{6}{\frac{33877}{3338}},\ \overset{20}{\frac{227528}{22419}}.$$

In La Grange's additions to Euler's Algebra it is shewn, that the integer values of x, which will make $103x^2 + 1$ a rational square, will be found amongst the denominators of the principal fractions converging to the value of $\surd 103$, and that the index or number written above the fraction will be 20, (twice the greatest integer contained in $\surd 103$) the last in the period of the continued fraction.

The first fraction of the kind which occurs is $\frac{227528}{22419}$, and upon trial the denominator will be found to succeed; the numerator 227528 being the root of $103x^2 + 1$, when $x = 22419$. And in like manner the next succeeding period of the principal converging fractions will give another answer.

SECOND SOLUTION, *by Mr.* JOHN BAINES, *Jun.*

Mathematical Master, at Mr. Reynard's Mathematical School, Reading.

Put $\surd(103x^2 + 1) = a$, and, neglecting + 1, as inconsiderable, we have $a = x\surd 103$, $\therefore \frac{a}{x} = \surd 103$, and $\frac{a}{x} > 10 < 11$. Put $a = 10x + b$, then $3x^2 - 20bx = b^2 - 1$, and $x = \frac{10b + \surd(103b^2 - 3)}{3} = \frac{10b + b\surd 103}{3}$ (nearly), $\therefore \frac{x}{b} = \frac{10 + \surd 103}{3}$, and $\frac{x}{b} > 6 < 7$. Put $x = 6b + c$, then $13b^2 - 16bc = 3c^2 + 1$, and $b = \frac{8c + \surd(103c^2 + 13)}{13} = \frac{8c + c\surd 103}{13}$, $\therefore \frac{b}{c} = \frac{8 + \surd 103}{13}$, and $\frac{b}{c} > 1 < 2$. Put $b = c + d$, then $6c^2 - 10cd = 13d^2 - 1$, and $c = \frac{5d + \surd(103d^2 - 6)}{6} = \frac{5d + d\surd 103}{6}$, $\therefore \frac{c}{d} = \frac{5 + \surd 103}{6}$, and $\frac{c}{d} > 2 < 3$. Put $c = 2d + e$, then $9d^2 - 14de = 6e^2 + 1$, and $d = \frac{7e + \surd(103e^2 + 9)}{9} = \frac{7e + e\surd 103}{9}$,

$\therefore \frac{d}{e} = \frac{7+\sqrt{103}}{9}$, and $\frac{d}{e} > 1 < 2$. Put $d = e + f$, then $11e^2 - 4ef = 9f^2 - 1$, and $e = \frac{2f + \sqrt{(103f^2 - 11)}}{11} = \frac{2f + f\sqrt{103}}{11}$, $\therefore \frac{e}{f} = \frac{2+\sqrt{103}}{11}$, and $\frac{e}{f} > 1 < 2$. Put $e = f + g$, then $2f^2 - 18fg = 11g^2 + 1$, and $f = \frac{9g + \sqrt{(103g^2 + 2)}}{2} = \frac{9g + g\sqrt{103}}{2}$, $\therefore \frac{f}{g} = \frac{9+\sqrt{103}}{2}$, and $\frac{f}{g} > 9 < 10$. Put $f = 9g + h$, then $11g^2 - 18gh = 2h^2 - 1$, and $g = \frac{9h + \sqrt{(103h^2 - 11)}}{11} = \frac{9h + h\sqrt{103}}{11}$, $\therefore \frac{g}{h} = \frac{9+\sqrt{103}}{11}$, and $\frac{g}{h} > 1 < 2$. Put $g = h + i$, then $9h^2 - 4hi = 10i^2 + 1$, and $h = \frac{2i + \sqrt{(103i^2 + 9)}}{9} = \frac{2i + i\sqrt{103}}{9}$, $\therefore \frac{h}{i} = \frac{2+\sqrt{103}}{9}$, and $\frac{h}{i} > 1 < 2$. Put $h = i + j$, then $6i^2 - 14ij = 9j^2 - 1$, and $i = \frac{7j + \sqrt{(103j^2 - 6)}}{6} = \frac{7j + j\sqrt{103}}{6}$, $\therefore \frac{i}{j} = \frac{7+\sqrt{103}}{6}$, and $\frac{i}{j} > 2 < 3$. Put $i = 2j + k$, then $13j^2 - 10jk = 6k^2 + 1$, and $j = \frac{5k + \sqrt{(103k^2 + 13)}}{13} = \frac{5k + k\sqrt{103}}{13}$, $\therefore \frac{j}{k} = \frac{5+\sqrt{103}}{13}$, and $\frac{j}{k} > 1 < 2$. Put $j = k + l$, then $3k^2 - 16kl = 13l^2 - 1$, and $k = \frac{8l + \sqrt{(103l^2 - 3)}}{3} = \frac{8l + l\sqrt{103}}{3}$, $\therefore \frac{k}{l} = \frac{8+\sqrt{103}}{3}$, and $\frac{k}{l} > 6 < 7$. Put $k = 6l + m$, then $l^2 - 20lm = 3m^2 + 1$, and $l = 10m + \sqrt{(103m^2 + 1)}$. Wherefore, having arrived at an equation similar to the original one, we may suppose $m = 0$, and $l = 1$, then

$6l + m = k = 6$
$k + l = j = 7$
$2j + k = i = 20$
$i + j = h = 27$
$h + i = g = 47$
$9g + h = f = 450$
$f + g = e = 497$
$e + f = d = 947$
$2d + e = c = 2391$,
$c + d = b = 3338$
$6b + c = x = 22419$, the No. req.
$10x + b = a = 227528$

Messrs. Adams, Dishneagh, *and* Hynes *answered it.*

III. QUESTION 313, *by Mr.* GEORGE HARVEY, Jun. *Plymouth.*

From a given right cone to cut such a parabolic section, that the solid generated by the revolution thereof, either about its axis, or its greatest ordinate may be the greatest possible.

FIRST SOLUTION, *by Mr.* CUNLIFFE.

Let ACB (fig. 112, pl. 4.) be the cone and EHF a parabolic section thereof. And first, to determine the section so that the solid generated by the revolution thereof about its axis GH may be the greatest possible. Put $n = \cdot 7854$, then it is well known that $4n \times GF^2 \times \frac{GH}{2} = 2n \times GF^2 \times GH$ expresses the content of the conoid generated by the revolution of the parabola about its axis GH. And because of the parallels GH, BC, $AB : BC :: AG : GH = (BC \div AB)\, AG$; therefore $2n \times GF^2 \times GH = 2n\,(BC \div AB) \times GF^2 \times AG$, which is to be a maximum, by the question; and therefore $GF^2 \times AG$ must be a maximum, because the factor $2n\,(BC \div AB)$ is a given quantity.

By attentively considering the form of the preceding expression, viz. $GF^2 \times AG$, it will be easily perceived that it amounts to the well known problem of determining the greatest cone which can be inscribed in a given sphere: wherefore $AG = \frac{2}{3} AB$.

Secondly. It is well known that the content of the solid generated by the revolution of the parabola about the ordinate GF is expressed by $\frac{32n}{15} \times GH^2 \times GF$; or, from what has been done in the former part, by $\frac{32n}{15} \times \frac{BC^2}{AB^2} \times AG^2 \times GF$, which is to be a maximum, by the question; therefore $AG^2 \times GF$ must be a maximum, and this is the case when $AG = \frac{5}{6} AB$, as will be found by making the fluxion of the expression $= 0$.

By attentively considering the expression $AG^2 \times GF$ it will be readily perceived that it may be turned into the following problem, viz. Having given the hypothenuse of a right-angled triangle, to determine the legs when the solid under the square of one of the segments of the hypothenuse made by a perpendicular from the right angle, and the said perpendicular is a maximum.

SECOND SOLUTION, *by Mr.* JOHN BAINES, Jun.

Let ACB (fig 113. pl. 4.) represent the given cone, FDG the required parabola, DE its axis, EG (perpendicular to AB) its greatest ordinate and $n =$ the area of a circle whose diameter is 1: Then the conoid or solid generated by the revolution of the curve about the axis ED is $= \frac{1}{2}$ its circumscribing cylinder $= \frac{1}{2}n \times \mathrm{FG}^2 \times \mathrm{ED}$; and the spindle or solid generated by its revolution about the ordinate FG is $= \frac{8}{15} n \times 4\,\mathrm{ED}^2 \times \mathrm{FG}$. Therefore in the first case $\mathrm{FG}^2 \times \mathrm{ED}$, and in the second $\mathrm{ED}^2 \times \mathrm{FG}$, or its square $\mathrm{ED}^4 \times \mathrm{FG}^2$ must be a maximum. Put $a = \mathrm{AB}$, $c = \mathrm{AC}$ and $x = \mathrm{BE}$; then by similar triangles $a : c :: x : \mathrm{ED} = \frac{cx}{a}$, and by the nature of the circle $\mathrm{FG}^2 = 4(a-x)x$; therefore $\mathrm{FG}^2 \times \mathrm{ED} = \frac{4c}{a}(a-x)\,x^2$, and leaving out the constant factor and taking the fluxion, $2ax\dot{x} - 3x^2\dot{x} = 0$; therefore $x = \frac{2}{3}a$. In the second case $\mathrm{ED}^4 \times \mathrm{FG}^2 = \frac{4c^4}{a^4}x^4 \times (ax - x^2) = \frac{4c^4}{a^4}x^5(a-x)$, therefore $5ax^4\dot{x} - 6x^5\dot{x} = 0$, and $x = \frac{5}{6}a$.

Solutions were received from Messrs. Adams, Dishneagh *and* Eratosthenes.

IV. QUESTION 314, *by Mr.* JOSEPH WILLIAMS.

Emerson in his Increments has made use of a series to determine the integral of $\frac{z-n}{rz}\underset{\cdot}{s} - \underset{\cdot}{s} = \underset{\cdot\cdot}{s}$. It is required to assign the integral in terms of s and $\underset{\cdot}{s}$ without using a series?

SOLUTION, *by Mr.* JOSEPH WILLIAMS, *the Proposer.*

$\frac{z-n}{rz}\underset{\cdot}{s} - \underset{\cdot}{s} = \underset{\cdot\cdot}{s}$, ∴ $\underset{\cdot}{s} = \frac{rz\underset{\cdot\cdot}{s}}{(1-r)z-n} = \frac{rz\underset{\cdot\cdot}{s}}{mz-n}$ (if we put $1-r=m$).

Let $\frac{rz}{mz-n} = v$, then $\underset{\cdot}{s} = v\underset{\cdot\cdot}{s}$, ∴ $s = v\underset{\cdot}{s} - \underset{1}{s}\underset{\cdot}{v} = s\underset{\cdot}{v} - \underset{\cdot}{v}s - \underset{\cdot}{v}\,\underset{\cdot}{s}$,

∴ $(1+\underset{\cdot}{v})s = (v-\underset{\cdot}{v})\underset{\cdot}{s}$, ∴ $s = \frac{v-\underset{\cdot}{v}}{1+\underset{\cdot}{v}}\underset{\cdot}{s}$.

Now $\underset{\cdot}{v} = \frac{rz_1}{mz_1 - n} - \frac{rz}{mz - n} = \frac{rn(z - z_1)}{(mz_1 - n)\,(mz - n)}$,

$\therefore\ v - \underset{\cdot}{v} = \frac{rz(mz_1 - n) + m\underset{\cdot}{z}}{(mz_1 - n)(mz - n)}$ and $1 + \underset{\cdot}{v} = \frac{(mz_1 - n)(mz - n) - rn\underset{\cdot}{z}}{mz_1 - n)(mz - n)}$.

$$\therefore\ s = \frac{rz\,(mz_1 - n) + rn\underset{\cdot}{z}}{(mz_1 - n)\,(mz - n) - rn\underset{\cdot}{z}}\underset{\cdot}{s},$$

$$\therefore\ s = \frac{rmzz_1 - rnz_{-1}}{m^2zz_1 - 2mnz - (m + r)\,n\underset{\cdot}{z} + n^2}\underset{\cdot}{s}.$$

Cor. Let $n = -n$, $r = 1$, $\therefore\ 1 - r = m = 0$,

and $s = \frac{nz_{-1}}{n\underset{\cdot}{z} + n^2}\underset{\cdot}{s} = \frac{z_{-1}\underset{\cdot}{s}}{n + \underset{\cdot}{z}}$ which is the same as that deduced by Emerson in a similar manner in his 23d Example.

V. QUESTION 315, *by* AMICUS.

If, from two given points, two straight lines be drawn to a point in a curve of any order, given by position, so that their rectangle may be the greatest or the least possible; these lines will have the same ratio as the cosines of the angles which they make with the curve (or with the tangent) at the point of intersection. Required the demonstration?

FIRST SOLUTION, *by Mr.* LOWRY.

Let A and B (fig. 114, pl. 4.) be the given points, EF the given curve; and suppose the lines AC, BC are drawn as required so that their rectangle is the greatest or least possible. Let *c* be a point in the curve indefinitely near to C, and draw A*c*, B*c*. With the centres A and B, and the distances AC, BC describe the arcs C*a*, C*b* to meet A*c* in *a* and B*c* in *b*; then *ac* is the increment of AC, and *bc* the decrement of BC; and the difference between AC × BC and A*c* × B*c* $\{= (\text{AC} + ac)(\text{BC} - bc)\}$, or the increase or decrease of the rectangle AC × BC, is AC × *bc* — BC × *ac* — *ac* × *cb*; that is, ultimately, AC + *bc* — BC × *ac*; and which in the case of a maximum or minimum is = 0; therefore AC : BC :: *ac* : *bc*. But since *c* is indefinitely near to C, the little triangles C*ca*, C*cb* may be considered as rectilineal and right-angled at a and b, therefore $ac = \text{C}c \times \cos \text{C}ca$ and $bc = \text{C}c \times \cos \text{C}cb$; and AC : BC :: cos C$ca$: cos Ccb (BcF). But the angles Cca and BcF are ultimately those which the lines AC, BC make with the curve, therefore the lines AC, BC are as the cosines of the angles which they make with the curve at the point C.

It may be proved in a similar manner that $\text{AC}^m \times \text{BC}^n$ is the greatest or least possible when $n \times \text{AC}$ and $m \times \text{BC}$ are as the cosines of the angles that AC and BC make with the curve.

It follows immediately from the above proposition that, if a polygon of any number of sides be inscribed in an oval figure, so that the sides are respectively proportional to the cosines of the angles which they make with the curve, then the continued product of all the sides of the ploygon will be greater than that of the sides of any other polygon of the same number of sides that can be inscribed in the given curve.

SECOND SOLUTION, *by Mr.* CUNLIFFE.

Let PQ (fig. 115, pl. 4.) represent the curve given by position, A and B the two given points. Suppose P the required point, p another point in the curve indefinitely near to the former, and draw the lines AP, BP, Ap, Bp. With the centres A and B and radii Ap, BP describe two small arcs cutting AP, Bp in r and s. Put AP $= x$, BP $= v$, then $ps = \dot{v}$ and $\text{P}r = -\dot{x}$. And by the principles of trigonometry $ps = \dot{v} : \text{P}r = -\dot{x}$ :: cos spP (BPQ) : cos APp. Now by the question the rectangle of the lines AP, BP or xv, is a maximum or minimum, and therefore its fluxion $x\dot{v} + v\dot{x} = 0$, that is $x\dot{v} = -v\dot{x}$; whence $\dot{v} : -\dot{x} :: v =$ BP : $x =$ AP : consequently BP : AP :: cos BPQ : cos APp.

A Solution was received from Mr. Adams.

VI. QUESTION 316, *by* G. V.

If from a given point straight lines be drawn to the extremities of any two conjugate diameters of an ellipse given by position, the sum of the squares of these lines is a given space. Required the demonstration?

SOLUTION, *by* ALIQUIS,

Let P (fig. 116, pl. 4.) be the given point, O the centre of the given ellipse, and AC and BD any two conjugate diameters; then if PA, PB, PC and PD be drawn, it is to be proved that $\text{AP}^2 + \text{BP}^2 + \text{CP}^2 + \text{DP}^2$ is a constant quantity.

Join PO; then, because AC and BD are bisected in O, $\text{AP}^2 +$

$PC^2 = 2AO^2 + 2PO^2$, and $PB^2 + PD^2 = 2BO^2 + 2PO^2$; therefore $AP^2 + PC^2 + PB^2 + PD^2 = 2(AO^2 + BO^2) + 4PO^2$ = a given space; for PO is a given line, and $AO^2 + BO^2$ is = the sum of the squares of the given semi-axes.

Cor. If lines be drawn from the extremities of any two conjugate diameters to touch a circle given in magnitude and position, the sum of the squares of the tangents will be a constant quantity.

Mr. Adams *sent a solution to this question.*

VII. QUESTION 317, *by* G. V.

Let ABDC be a parallelogram, draw the diagonal BC, and draw DE perpendicular to BC; then, perpendiculars drawn to AB, AC at the points B and C shall intersect each other in the line DE. Required the demonstration?

First Solution, *by Mr.* John Dawes, *Birmingham.*

Let ABCD (fig. 117, pl. 4.) be the parallelogram, F the intersection of the perpendiculars from AB and AC; then because the angles ABF, ACF are right angles, therefore a circle passes through the points A, B, C, F, of which AF is the diameter; join FL and BL (L being the point where the circle cuts CD again), then BL = AC = BD, therefore the triangle BLD is isosceles: but BF cuts CD at right angles, in M; therefore LM = MD, and the triangle LFD is isosceles, and ∠ BFL = ∠ BFH, therefore BL = BH = AC. Let FH intersect AB in K, now because KBF is a right angle, and KBE (= ABC) = KFB, therefore FEB is a right angle, therefore DE meets the diagonal BC at right angles, and therefore the point F is in the line DE.

Second Solution, *by* Eratosthenes.

Let the perpendiculars at B and C meet in F (fig. 118, pl. 4.) and join DF, then it is to be proved that the points E, D, F are in a straight line. Produce BD to meet CF in H and CD to meet BF in G, and join GH; then, bcause of the parallels, BH will be perpendicular to CF and CG to BF, therefore the points B, G, H, C will be in a circle, the diameter of which is BC, and the angle HGF = BCH; also the points G, D, H, F will be in a circle, the diameter of which is DF, and the angle HDF = HGF, therefore the angle BCH = angle HDF: but ∠ BCH + ∠ EDH = two right angles; therefore ∠ HDF + ∠ EDH = two right angles, and the points E, D, F are in a straight line.

Ingenious demonstrations were received from Messrs. Adams, Baines, *and a* Lady.

VIII. QUESTION 318, *by* G. V.

If from one of the angles of a rectangle a perpendicular be drawn to its diagonal, and from their intersection lines be drawn perpendicular to the sides containing the opposite angle, then, putting P and p for these last perpendiculars, and D for the diagonal, $\mathrm{P}^{\frac{2}{3}} + p^{\frac{2}{3}} = \mathrm{D}^{\frac{2}{3}}$. Required the demonstration?

First Solution, *by Mr.* R. J. Dishneagh, *Trinity College, Cambridge.*

Fig. 119. pl, 4. Let $\angle\ \mathrm{FBE} = \theta$, rad 1. $\therefore\ \mathrm{PE} = \mathrm{EB} \,.\, \sin\theta$, but $\mathrm{EB} = \mathrm{BD} \,.\, \cos\ \mathrm{EBD} = \mathrm{BD} \,.\, \sin\theta$, $\therefore\ \mathrm{FE} = \mathrm{BD} \,.\, \sin^2\theta$: but $\mathrm{BD} = \mathrm{CD} \,.\, \sin\ \mathrm{BCD} = \mathrm{CD} \,.\, \sin\theta$, $\therefore\ \mathrm{FE} = \mathrm{CD} \,.\, \sin^3\theta$, or in the notation of the question $\mathrm{P} = \mathrm{D} \,.\, \sin^3\theta$. In the same way

$$p = \mathrm{D} \,.\, \cos^3\theta. \quad \text{Hence} \left.\begin{array}{r} \mathrm{P}^{\frac{2}{3}} = \mathrm{D}^{\frac{2}{3}} \,.\, \sin^2\theta \\ p^{\frac{2}{3}} = \mathrm{D}^{\frac{2}{3}} \,.\, \cos^2\theta \end{array}\right\} \therefore\ \mathrm{P}^{\frac{2}{3}} + p^{\frac{2}{3}} = \mathrm{D}^{\frac{2}{3}}.$$

Second Solution, *by Mr.* Jas. Adams, *Stonehouse.*

By similar triangles (fig. 120. pl. 4.)

$$\left.\begin{array}{l} \mathrm{AC} : \mathrm{CD} :: \mathrm{AE} : \mathrm{EG} \\ \mathrm{AC} : \mathrm{AD} :: \mathrm{EC} : \mathrm{FF} \end{array}\right\} \text{therefore} \left\{\begin{array}{l} \mathrm{AC}^2 : \mathrm{AC} \,.\, \mathrm{AE} :: \mathrm{CD} : \mathrm{EG} \\ \mathrm{AC}^2 : \mathrm{AC} \,.\, \mathrm{EC} :: \mathrm{AD} : \mathrm{EF}. \end{array}\right.$$

But $\mathrm{AC} \,.\, \mathrm{AE} = \mathrm{AB}^2 = \mathrm{CD}^2$ and $\mathrm{AC} \,.\, \mathrm{EC} = \mathrm{BC}^2 = \mathrm{AD}^2$.

Therefore $\mathrm{AC}^2 : \mathrm{CD}^2 :: \mathrm{CD} : \mathrm{EG}$ and $\mathrm{AC}^2 : \mathrm{AD}^2 :: \mathrm{AD} : \mathrm{EF}$.

Theref. $\mathrm{EG} = \dfrac{\mathrm{CD}^3}{\mathrm{AC}^2}$ and $\mathrm{EF} = \dfrac{\mathrm{AD}^3}{\mathrm{AC}^2}$: or $\mathrm{EG}^{\frac{2}{3}} = \dfrac{\mathrm{CD}^2}{\mathrm{AC}^{\frac{4}{3}}}$ and $\mathrm{EF}^{\frac{2}{3}} = \dfrac{\mathrm{AD}^2}{\mathrm{AC}^{\frac{4}{3}}}$;

Hence $\mathrm{EG}^{\frac{2}{3}} + \mathrm{EF}^{\frac{2}{3}} = \dfrac{\mathrm{AD}^2 + \mathrm{CD}^2}{\mathrm{AC}^{\frac{4}{3}}} = \dfrac{\mathrm{AC}^2}{\mathrm{AC}^{\frac{4}{3}}} = \mathrm{AC}^{2-\frac{4}{3}} = \mathrm{AC}^{\frac{2}{3}}$.

Demonstrations were received from Messrs. Baines *and* Eratosthenes.

IX. QUESTION 319, *by Mr.* Cunliffe, *R. M. College.*

Suppose a projectile delivered with a given velocity, what must be the angle of elevation that the length of the curve described in its flight above an horizontal plane may be the greatest possible, supposing no resistance from the air?

First Solution, *by Mr.* Cunliffe, *the Proposer.*

It is known that a projectile would describe the curve of a a conic parabola, if not resisted by the air. In order therefore to render the solution more complete I shall begin with finding the length of the arc of a parabola. The equation of the curve is $2px = y^2$; whence $\dot{x} = \frac{y\dot{y}}{p}$ and $\sqrt{(\dot{x}^2+\dot{y}^2)} = \frac{\dot{y}}{p}\sqrt{(p^2+y^2)} =$ the fluxion of an arc of the curve; and the correct fluent is $\frac{y}{2p}\sqrt{(p^2+y^2)} + \frac{p}{2} \times$ h. l. $\frac{y+\sqrt{(p^2+y^2)}}{p}$, which expresses the length of an arc of the parabola intercepted between the principal vertex and the rectangular ordinate y; and therefore $\frac{y}{p}\sqrt{(p^2+y^2)} + p \times$ h. l. $\frac{y+\sqrt{(p^2+y^2)}}{p}$ expresses the length of the arc cut off by a double ordinate.

Now to apply what has been done to the solution of the question: Let s denote the sine of the angle of elevation of the projectile, c its cosine, to radius 1; also let a denote the height due to the given projectile velocity, that is the height whence a heavy body must freely fall from rest, to acquire that velocity. Then it is shewn by the writers on gunnery, that as^2 expresses the greatest height of the projectile, or the abscissa of the parabolic arc which it describes. Also $4asc$ expresses the horizontal range or double ordinate of the curve described by the projectile in its flight*. Wherefore put $x = as^2$ and $y = 2asc$, then $p = \frac{y^2}{2x} = 2ac^2$, and hence $\sqrt{(p^2+y^2)} = 2ac$; $\frac{y}{p}\sqrt{(p^2+y^2)} = 2as$; $\frac{y+\sqrt{(p^2+y^2)}}{p} = \frac{1+s}{c} = \sqrt{\frac{1+s}{1-s}}$; and by means of these the expression $\frac{y}{p}\sqrt{(p^2+y^2)} + p \times$ h. l. $\frac{y+\sqrt{(p^2+y^2)}}{p}$ becomes $2as + 2ac^2 \times$ h. l. $\sqrt{\frac{1+s}{1-s}} = 2as + ac^2 \times$ h. l. $\frac{1+s}{1-s}$, which is to be a maximum by the question, therefore $2s + c^2 \times$ h. l. $\frac{1+s}{1-s} = 2s + (1-s^2) \times$ h. l. $\frac{1+s}{1-s}$ must be a maximum, because a is given. Therefore putting the fluxion of the last expression $= 0$, $4\dot{s} - 2s\dot{s} \times$ h. l. $\frac{1+s}{1-s} = 0$, which gives $s \times$ h. l. $\frac{1+s}{1-s} = 2$; and hence, by the method of trial and error, $s = \cdot 8335565$, the sine of 56°. 27′. 56″ 97, the required angle of elevation.

---

* See Art. 334, vol. 2d. Dalby's Course of Mathematics.

SECOND SOLUTION, *by Mr.* R. J. DISHNEAGH.

It is well known that the equation to any parabola described by a projectile is $y = x\dfrac{\sin\theta}{\cos\theta} - \dfrac{x^2}{4h\cos^2\theta}$ where x and y denote the rectangular cordinates whose common origin is the point of projection. Putting $y = 0$, it is easily seen that the two corresponding values of x are 0, and $4h \,.\, \sin\theta \,.\, \cos\theta$. Now $\dot{y} = \dot{x}\left(\dfrac{\sin\theta}{\cos\theta} - \dfrac{x}{2h\cos^2\theta}\right) \therefore \dot{z}^2 = \dot{x}^2 + \dot{y}^2 =$ $\dot{x}^2\left\{1 + \left(\dfrac{\sin^2\theta}{\cos^2\theta} - \dfrac{x}{2h\cos^2\theta}\right)^2\right\} = \dfrac{\dot{x}^2}{4h^2\cos^4\theta}(4h^2\cos^2\theta - 4h \,.\, \cos\theta \,.\, \sin\theta \,.\, x + x^2)$ after proper reduction, $\therefore \dot{z} = \dfrac{\dot{x}}{2h\cos^2\theta} \times \surd(4h^2\cos^2\theta - 4h\cos\theta \,.\, \sin\theta \,.\, x + x^2)$.

Put $v = x - 2h\sin\theta \,.\, \cos\theta$, or for greater simplicity $= x - 2hsc$, $\therefore v^2 = x^2 - 4hscx + 4h^2s^2c^2$, hence, by substitution $\dot{z} = \dfrac{\dot{v}}{2hc^2}\surd(4h^2c^2 - 4h^2s^2c^2 + v^2) = \dfrac{\dot{v}}{2hc^2}\surd\left\{4h^2c^2(1-s^2)+v^2\right\}$ $= \dfrac{\dot{v}}{2hc^2}\surd(4h^2c^4+v^2)$ which expression is to be integrated between the values of $x = 0$ and $x = 4hsc$, that is between the vàlue of $v = -2hsc$ and $v = 2hsc$.

Now $z = \dfrac{1}{4hc^2} \times v\surd(4h^2v^4+v^2) + hc^2 \,.\, \log\colon v + \surd(4h^2c^4+v^2)$ (see Vince's Fluxions, p. 86, 3d. edit.) $\therefore$ putting $v = -2hsc$, we have, after proper reduction, $z = -hs + hc^2 \log. 2hc(1-s)$ and putting $v = 2hsc$, $z = hs + hc^2 \log. 2hc(1+s)$.

Now the real value of z for the whole curve is = the difference of these two values of z,

$$\therefore \text{ whole length } = 2hs + hc^2 \,.\, \log. \frac{1+s}{1-s};$$

$$\therefore \text{ by the question, } 2s + c^2 \,.\, \log. \frac{1+s}{1-s} = \text{max.}$$

$$\text{But } \frac{1+s}{1-s} = \frac{1+\sin\theta}{1-\sin\theta} = \tan^2\left(\frac{\pi+\theta}{2}\right),\ \pi = 90°,$$

$$\therefore 2\sin\theta + \cos^2\theta \,.\, \log. \tan^2\left(\frac{\pi+\theta}{2}\right) = \text{max.}$$

$$\therefore \sin\theta + \cos^2\theta \,.\, \text{log. tan}\,\frac{\pi+\theta}{2} = \text{max.}$$

Now the fluxion of $\tan\dfrac{\pi+\theta}{2} = \dfrac{\dot\theta}{2} \times \dfrac{1}{\cos^2\left(\dfrac{\pi+\theta}{2}\right)}$

$$\therefore \text{ fluxion of log. tan}\,\frac{\pi+\theta}{2} = \frac{\dot\theta}{2\,.\sin\dfrac{\pi+\theta}{2} \times \cos\dfrac{\pi+\theta}{2}} =$$

$\dfrac{\dot\theta}{\sin(\pi+\theta)} = \dfrac{\dot\theta}{\sin(\pi-\theta)} = \dfrac{\dot\theta}{\cos\theta}$; ∴ taking the fluxions

$$\dot\theta\cos\theta - 2\dot\theta\sin\theta\,.\cos\theta\,.\,\text{log. tan}\,\frac{\pi+\theta}{2} + \cos^2\theta \times \frac{\dot\theta}{\cos\theta} = 0.$$

Hence $1 - 2\sin\theta\,.\,\text{log. tan}\,\dfrac{\pi+\theta}{2} + 1 = 0.$

$$\therefore \text{Hyp. log. tan}\,\frac{\pi+\theta}{2} = \frac{1}{\sin\theta} = \text{cosec}\,\theta;$$

where θ is to be determined by trial from the tables.

X. QUESTION 320, *by Mr.* CUNLIFFE.

In a given circle to inscribe a triangle having two of its sides in a given ratio, and the sum of the other side and perpendicular thereon from the opposite angle a maximum.

FIRST SOLUTION *by Mr.* LOWRY.

It does not appear that this problem can be constructed by plane geometry, except in some particular cases, but it may be done generally, when the sum is a constant quantity, by means of the ellipse. Let ABC (fig. 121. pl. 4.) be the triangle required, and BK the diameter of the circumscribing circle, and draw AK and CK to meet the perpendicular BH produced in D and S; then because the angle ABK = ACK = SBC, and the angle KBC = KAC = ABD, the triangles BCS and ADB are respectively similar to ABK and CBK; therefore AB : BK :: BC : BS, and BC : BK :: AB : BD. But, by the question, AB has to BC a given ratio, and BK is a given line, therefore BS and BD are also given lines; and because BAD and BCS are right angles, the semicircles described on the diameters BD and BS are the loci of the points A and C, and we have only to draw AC perpendicular to BD (fig. 122. pl. 4,) so that the sum of BH and AC may be either a constant quantity, or the greatest possible. Let O be the centre of the circle BAD, and join AO: then because BO is a given line, the sum of OH and AC will be either a given quantity or a maximum. Draw IO to make the angle $IOH =$ half a right angle, and produce HC till CN be $= AH$, and draw NL parallel to IO, meeting the perpendiculars at B and D in L and M, and OP parallel to IN, meeting NL in P; then IH will be $= OH$, and $IN = OH + AC =$ a given line; therefore the line NL will be given by position, and P will be a given point. Also $CN^2 = AH^2 = AO^2 - OH^2$, and since $PN^2 = IO^2 = 2OH^2$, and $PL^2 = 2DB^2 = 2AO^2$, $2CN^2 = 2AO^2 - PN^2 = PL^2 - PN^2$, which is the property of the ellipse whose centre is P, semi-diameter PL or $PM = AO \times \surd 2$, and semi-conjugate to $PL = AO$. Hence if an ellipse be described from the centre P, with the semi-diameters $PL = AO \times \surd 2$ and $PQ = AO$, it will intersect the circle BCS in the point C required.

The maximum will evidently obtain when the ellipse touches the circle BCS, or when the radius O′C is a normal to the ellipse at C, and in this case it follows, from well known properties of the curves, that $BH \cdot HN = AO \cdot HC + O'C \,.\, CN + HC \,.\, CN$, or which is the same thing $BH \,.\, AC = AO \,.\, HC + O'C \,.\, AH + HC \,.\, HA$.

When the given ratio is that of equality, BS and BD are each equal to the diameter BK: In this case $CN = CH$, and the locus of the point C will be a straight line drawn from the intersection of the lines SB, PL to the middle of OP, and in the maximum case the above expression becomes $2BH \,.\, HC = 2AO \,.\, HC + HC^2$, or $2BH = 2OA + HC$; that is $2OH = HC$, which we know to be true from other principles.

SECOND SOLUTION, *by* Mr. CUNLIFFE, *the Proposer*.

Generally, when the sum of the side and perpendicular thereon from the opposite angle is equal to a given line s.

Let ACB (fig. 123. pl. 4.) represent the required triangle inscribed in the given circle, EF being a diameter bisecting the side AB in M. Draw CH parallel to AB meeting EF in H; and let AC, BC be the two sides having the given ratio; then their sum and difference will have a given ratio, or the square of their sum will have a given ratio to the square of their difference, and let this ratio be denoted by that of m to n.

Put $EF = d$, $EH = z$, and $EM = x$; also put $AB + MH = 2AM + MH = s$. Then $HF = d - z$, $FM = d - x$, $MH = z - x$, and by the property of the circle $AM = MB = \surd(dx - xx)$; whence $2AM + MH = 2\surd(dx - xx) + z - x = s$; also, we have, by known properties, $(AC + BC)^2 = 4EH \times MF = 4z(d - x)$, and $(AC - BC)^2 = 4EM \times HF = 4x(d - z)$. Again, by the question, $(AC + BC)^2 : (AC - BC)^2 :: m : n$, whence $n\,(AC + BC)^2$

$= m\,(\mathrm{AC}-\mathrm{BC})^2$, that is, $4nz\,(d-x) = 4mx\,(d-z)$, or $nz(d-x) = mx(d-z)$ therefore $z = \frac{mdx}{nd+(m-n)x} = \frac{mdx}{m-n} \div \left(\frac{nd}{m-n} + x\right) = \frac{ax}{b+x}$, by putting $a = \frac{md}{m-n}$, and $b = \frac{nd}{m-n}$. By means of the above value of z, $2\sqrt{(dx-xx)}+z-x = 2\sqrt{(dx-xx)} + \frac{ax}{b+x} - x = s$, which, being cleared of the surd, becomes $4(b+x)^2 \times (dx-xx) = (bs+(b+s+a)x+xx)^2$ which is a biquadratic equation, from whence x is to be found.

Again $2\sqrt{(dx-xx)} + z - x = 2\sqrt{(dx-xx)} + \frac{ax}{b+x} - x = s$, the sum of the base and perpendicular; and putting the fluxion of this equation $= 0$, we shall from thence obtain an equation from whence x may be found when the sum of the base and perpendicular is a maximum.

XI. QUESTION 321, *by Mr.* CUNLIFFE.

Suppose a piece of cast iron in the form of a cone, the diameter of the base being two, and the perpendicular height six feet. Suppose the cone suspended by the vertex, about which it is freely moveable :- let two weights, of half a ton each, be fastened by cords to a ring in the circumference of the base, and let the cords pass freely over two pullies in the same horizontal line with the vertex, at the distance of four and five feet, on the same side of it. Required the position of the cone's axis when in equilibrio?

FIRST SOLUTION, *by Mr.* CUNLIFFE, *the Proposer.*

Let VRR′ (fig. 124. pl. 4.) represent the cone suspended by the vertex V, and let VMN be an horizontal line passing through V, in which are situated the two points or pullies M, N at the respective distances of four and five feet from the vertex V.

Let P and Q represent the two weights fastened by the cords PNR, QMR to a ring R, in the circumference of the cone's base. In VX, the axis of the cone, take $\mathrm{VG} = \frac{3}{4}\,\mathrm{VX}$, then G will be the centre of gravity of the cone: draw GD perpendicular to VMN; also draw NK, ML at right angles to VR, and let W denote the weight of the cone. Then by the known principles of mechanics $(\mathrm{VD} \div \mathrm{RV}) \times \mathrm{W}$ will express the force which acting at R, in a direction perpendicular to RV, would support the cone in that position. Again by the resolution of forces $(\mathrm{NK} \div \mathrm{NR}) \times \mathrm{P} =$

$P = P \times \sin \angle VRN$ will express the effect of the weight P at R, in a direction perpendicular to RV; and for the same reason $Q \times \sin \angle VRM$ will express the effect of the weight Q at R, in a direction perpendicular to RV. And therefore when the cone and weights are in equilibrio.

$P \times \sin \angle VRN + Q \times \sin \angle VRM = (VD \div RV) \times W$, or $RV \times (P \times \sin \angle VRN + Q \times \sin VRM) = VD \times W$.

Put $VX = a$, $XR = b$, $VM = m$, $VN = n$, $RV = l$, sine $\angle RVM = s$, its cosine $= c$; radius 1. Then by known properties $RM^2 = RV^2 + MV^2 - 2c \times RV \times MV = l^2 + m^2 - 2lmc$, and $RN^2 = l^2 + n^2 - 2lnc$, $RN : NV :: s : \sin \angle VRN$; whence $\sin \angle VRN = (NV \div RN) \times s = ns \div \sqrt{(l^2 + n^2 - 2lnc)}$. And in like manner we find $\sin \angle VRM = (MV \div RM) \times s = ms \div \sqrt{(l^2 + m^2 - 2lmc)}$.

Again, by the principles of trigonometry, $\sin \angle RVX = b \div l$, its cosine $= a \div l$; hence, and by a known theorem, $\sin \angle GVD = (as + bc) \div l$, and its cosine $= (ac - bs) \div l$, which is the sine of the $\angle VGD$: Also $VD = VG \times \sin \angle VGD = 3a(ac - bs) \div 4l$. And by means of these the expression

$$RV \times (P \times \sin \angle VRN + Q \times \sin \angle VRM) = VD \times W$$

becomes $l \times \left(\frac{P \times ns}{\sqrt{(l^2 + n^2 - 2lnc)}} + \frac{Q \times ms}{\sqrt{(l^2 + m^2 - 2lmc)}}\right) = \frac{3aW}{4} \times \frac{ac - bs}{l}$,

or $4l^2 \times \left(\frac{P \times ns}{\sqrt{(l^2 + n^2 - 2lnc)}} + \frac{Q \times ms}{\sqrt{(l^2 + m^2 - 2lmc)}}\right) = 3aW(ac - bs)$,

from which equation s may be found.

SECOND SOLUTION, *by* ERATOSTHENES.

Let CPL (fig. 125. pl. 4.) be the cone: Then from the given dimensions, the weight, the centre of gravity and the other dimensions may be found.

Let B and D be the two pullies, draw KE, APS perpendicular to CAD and join EP; let $CB = a$, $CD = b$, $CP = c$, $CE = e$, $\angle ACP = \phi$, $\angle ECP = \alpha$, $W =$ weight of cone, $w =$ weight at R, and $w' =$ weight at V; then $\angle KCE = \phi + \alpha$, and $\sin CSP : \sin CPS :: CP : CS$; but $\sin CSP = \cos SCA = \cos(\phi + \alpha)$ and $\sin CPS = \sin CPA = \cos PCA = \cos \phi$; therefore $\cos(\phi + \alpha) : \cos \phi :: c : CS = c\,\frac{\cos \phi}{\cos(\phi + \alpha)}$.

$W : \text{wt. at } S :: CS : CE \;\therefore\; \text{wt. at } S = \frac{e}{c} \times \frac{\cos(\phi + \alpha)}{\cos \phi} \times W.$

$CA = e \cos \phi$, $AP = c \sin \phi$, $AB = a - c \sin \phi$, $PB = \sqrt{(c^2 \sin^2 \phi + a^2 - 2ac \cos \phi + c^2 \cos^2 \phi)} = \sqrt{(a^2 + c^2 - 2ac \cos \phi)}$, $DP = \sqrt{(b^2 + c^2 - 2bc \cos \phi)}$.

wt. in direc. PB : wt. in direc. PA :: PB : AP.

$$\therefore \text{ wt. in direc. PA} = \frac{c \sin\varphi}{\sqrt{(a^2+c^2-2ac\cos\varphi)}} \times \text{wt. from } w.$$

wt. in direc. PB : wt. in direc. PA :: PD : PA,

$$\therefore \text{ wt. in direc. PA} = \frac{c \sin\varphi}{\sqrt{(b^2+c^2-2bc\cos\varphi)}} \times \text{wt. from } w'.$$

That the cone may be at rest the sum of the pressures upwards must equal its effective weight, therefore

$$\frac{wc\sin\varphi}{\sqrt{(a^2+c^2-2ac\cos\varphi)}} + \frac{wc'\sin\varphi}{\sqrt{(b^2+c^2-2bc\cos\varphi)}} = \frac{e}{c}\, w \times \frac{\cos(\varphi+\alpha)}{\cos\varphi},$$

from which equation the value of the angle φ may be determined.

THIRD SOLUTION, *by Mr.* LOWRY.

Let ABC (fig. 126. pl. 4.) represent the cone, G its centre of gravity, and E, F the pullies in the horizontal line AK. Draw GD and BP perpendicular to AK, and put $a =$ AB, $b =$ AE, $c =$ AF, $d =$ AG $= \frac{3}{4} \times 6 = 4\frac{1}{2}$, W = the weight of the cone, w and w' the weights suspended from E and F respectively, $\varphi =$ the angle BAP, and $e =$ the given angle BAG: Then, by the resolution of forces, the weights w and w' acting in the directions BE and BF are equivalent to the forces $w \times \sin$ BEP and $w' \times \sin$ BFP respectively, acting in the direction BP; and therefore the whole effect of the weights in supporting the cone, is equivalent to $w \times \sin$ BEP $+ w' \times \sin$ BFP acting in the vertical direction BP. Hence, when the whole is in equilibrio, we have, by the principles of statics, W × AD = ($w \times \sin$ BEP $+ w' \times \sin$ BFP) × AP,

$$\text{or W} \times \frac{\text{AD}}{\text{AP}} = w \times \sin \text{BEP} + w' \times \sin \text{BFP}.$$

But by trigonometry $\sin \text{BEP} = \sin \text{BEA} = \frac{a\sin\varphi}{\sqrt{(a^2+b^2-2ab\cos\varphi)}}$,

$\sin \text{BFP} = \sin \text{BFA} = \frac{a\sin\varphi}{\sqrt{(a^2+c^2-2ac\cos\varphi)}}$, $\text{AD} = d\cos(\varphi+e)$

and $\text{AP} = a\cos\varphi$, whence by substitution

$$\text{W} \times \frac{d\cos(\varphi+e)}{\cos\varphi} = \frac{w \times a\sin\varphi}{\sqrt{(a^2+b^2-2ab\cos\varphi)}} + \frac{w' \times a\sin\varphi}{\sqrt{(a^2+c^2-2ac\cos\varphi)}},$$

and from this equation the value of φ may be found by the method of trial and error, or by the common resolution of equations by approximation.

XII. QUESTION 322, *by Mr.* LOWRY, *R. M. College.*

To find the nature of the curve which shall cut all the parabolas that can be described by a projectile, thrown from a given point, with a given velocity, so that the time of flight, from the given point to the curve, may be the same in each parabola.

FIRST SOLUTION, *by Mr.* R. J. DISHNEAGH.

Let h = the space to acquire the velocity of projection,
p = space fallen through by gravity in given time.

AN (fig. 127. pl. 4.) $= x$, PN $= y$, AM being any direction of projection and = the space which would be described in the given time, the velocity of projection being continued uniform.

∴ AM = vel. × time $= \sqrt{(4mh)} \times \sqrt{(p \div m)}$, $m = 16\frac{1}{12}$ feet $= \sqrt{(4hp)}$. Take MP $= p$, and P is evidently a point in the parabola described, and also in the new curve. But MN^2 + AN^2 = AM^2, therefore $(y + p)^2 + x^2 = 4hp$ which is the equation to a circle whose rad. $= \sqrt{(4hp)}$ = AM, and whose axis lies at the distance p below AN and parallel to it as in the figure.

Observation. The problem may be solved upon a similar principle when the force varies as any function of the distance from the horizontal plane.

SECOND SOLUTION, *by Mr.* LOWRY.

Let IK (fig. 128. pl. 4.) be the curve required, AGI a portion of one of the parabolas described by the projectile, AD the direction, and DIH a vertical line passing through I and meeting the horizontal plane AK in H. Then the time of describing the curve AGI is evidently the same as that in which AD would be described uniformly with the given velocity, and also equal to the time, in which DI would be described by the action of gravity; therefore because the time of flight is to be the same in each parabola, AD and DI must be of the same lengths for every angle of elevation.

Put a = the given velocity per second, t = the time, $g = 16\frac{1}{12}$ feet, $x =$ AH, and $y =$ HI, then AD is $= at$, and DI $= gt^2$; but AH^2 + HD^2 = AD^2, therefore $x^2 + (y + gt^2)^2 = a^2t^2$, which is the equation to a circle whose radius is $= at$ and distance of the centre O from AK $= gt^2$, AO being perpendicular to AK.

XIII. QUESTION 323, *by Mr.* LOWRY.

To find curves such, that any straight line being drawn through a given point to meet the curve in two points, the rectangle under it and a given line may be equal to the rectangle of the segments intercepted between the given point and the curve.

FIRST SOLUTION, *by* A. B.

Fig. 129. pl. 4. Let QP $= \rho$, PR $= \rho'$; then the property which the curve is required to have is expressed by this equation, viz.

$$\rho\rho' = p(\rho + \rho'),$$

whence the equation following is derived, viz.

$$\frac{1}{p} = \frac{1}{\rho} + \frac{1}{\rho'}:$$

Now $\frac{1}{\rho}$ and $\frac{1}{\rho'}$ are to be considered as functions of one variable quantity, depending upon the position of QR with regard to the axis; and the most proper quantity for this purpose, is the ratio of the ordinates, $\frac{\text{QS}}{\text{SP}}$, or $\frac{\text{TR}}{\text{TP}}$ (denoted by u) which is the same at the opposite points of the curve Q and R: and as a line drawn through P is required to cut the curve in two points, we must choose such a function of u as is susceptible of two values; which will be accomplished by putting

$$\frac{1}{\rho} = fu + \surd(\text{F}u),$$

$$\frac{1}{\rho'} = fu - \surd(\text{F}u),$$

whence $\frac{1}{\rho} + \frac{1}{\rho'} = 2fu$; and this being compared with the preceding equation, we get $\frac{1}{p} = 2 . fu$; consequently the nature of the curves in question is thus expressed,

$$\frac{1}{\rho} = \frac{1}{2p} \pm \surd(\text{F}u)$$

where Fu denotes any function of u, or of $\frac{x}{y} = \frac{\text{QS}}{\text{SP}}$.

SECOND SOLUTION, *by* ERATOSTHENES.

Let PAQ (fig. 130. pl. 4.) be the required curve, $SP = y$, $SQ = y'$, $\angle PSA = \varphi$, $y = F(\varphi)$, then $y' = F(\varphi + \pi)$ and $F(\varphi) \times F(\varphi+\pi) = c \times (F(\varphi) + F(\varphi+\pi))$; let $F(\varphi) = \frac{1}{A + Bf(\varphi)}$, then $F(\varphi + \pi) = \frac{1}{A + Bf(\varphi + \pi)}$ and $\frac{1}{(A + Bf(\varphi)(A + Bf(\varphi + \pi))} = \frac{c}{A + Bf(\varphi)} + \frac{c}{A + Bf(\varphi + \pi)} = \frac{(cA + cA + cB(f(\varphi) + (\varphi + \pi))}{(A + Bf(\varphi))(A + Bf(\varphi + \pi))}$. That this equation may subsist the function denoted by f must change from positive to negative when φ becomes $\varphi + \pi$ and *vice versa*, and also $1 - 2cA = 0$, or $A = \frac{1}{2c}$; therefore $y = \frac{1}{\frac{1}{2c} + Bf(\varphi)} = \frac{2c}{1 + 2cBf(\varphi)}$ is the equation of the curve; if $B = \frac{1}{2c}$ and $f(\varphi) = -f(\varphi + \pi) = \cos\varphi$, $y = \frac{2c}{1 + \cos\varphi}$ which is an equation to the conic sections.

XIV. QUESTION 324, *by* Mr. LOWRY.

With what radius must a circle be described, from a given point as a centre, so that intersecting another circle given by position, the length of the arc intercepted by the given circle may be a maximum?

FIRST SOLUTION, *by* Mr. CUNLIFFE.

Let P (fig. 131. pl. 4.) be the given point, and ARB the given circle and through the centre C draw the right line PAB. Let PR be the radius of the required circle, and with the centre P describe the arc RS cutting AB in S, and draw CM perpendicular to PR; also draw the radius CR. Put $CR = r$, $PC = a$, $PR = v$; and let z denote the arc of a circle radius 1, and centre P, intercepted between the lines PR, PB; also let x denote the sine of z. Then it is obvious that $1 : z :: PR = v : RS = vz$, and when the arc RS is a maximum its fluxion $= 0$; that is $v\dot{z} + z\dot{v} = 0$, or $z = -\frac{v\dot{z}}{\dot{v}}$. By the principles of trigonometry $1 : x :: PC = a : CM = ax$; hence $PM^2 = PC^2 - CM^2$

$= a^2 - a^2x^2 = a^2(1 - x^2)$, or PM $= a\sqrt{(1 - x^2)}$, and MR$^2 =$ CR^2 − CM$^2 = r^2 - a^2x^2$, or MR $= \sqrt{(r^2 - a^2x^2)}$: also PR $=$ PM + MR, that is $v = a\sqrt{(1 - x^2)} + \sqrt{(r^2 - a^2x^2)}$. From the last equation $\dot{v} = -\frac{ax\dot{x}}{\sqrt{(1 - x^2)}} - \frac{a^2x\dot{x}}{\sqrt{(r^2 - a^2x^2)}}$; and it is known that $\dot{z} = \frac{\dot{x}}{\sqrt{(1 - x^2)}}$; whence the expression $z = -\frac{v\dot{z}}{\dot{v}}$ becomes $= \frac{a\sqrt{(1-x^2)}\sqrt{(r^2-a^2x^2)} + r^2 - a^2x^2}{ax\sqrt{(r^2-a^2x^2)} + a^2x\sqrt{(1-x^2)}} = \frac{\sqrt{(r^2 - a^2x^2)}}{ax}$ $=$ tan ∠MCR; from which equation x may be found either by the method of trial and error, or the reversion of series.

Corol. When $a = r$, or the point P falls in the periphery of the given circle, the preceding expression becomes $z = \frac{\sqrt{(1 - x^2)}}{x}$ = cotan. of ∠CPR; and the question is then reduced to finding the arc of a circle whose length shall be equal to the length of its cotangent.

SECOND SOLUTION *by Mr.* LOWRY.

Let ABC (fig. 132. pl. 4.) be the given circle, O its centre, P the given point, and BHK the arc required. Draw BO and BP, and put PO $= a$, radius BO $= r$, PB $= z$, and the measure of the angle BPH, radius 1, $= \varphi$: Then $1 : \varphi :: z :$ arc BH $= \frac{1}{2}$ BK $= \varphi z$; therefore φz must be a maximum, and consequently its fluxion $\dot{\varphi}z + \varphi\dot{z} = 0$, or $\frac{-\dot{\varphi}}{\dot{z}} = \frac{\varphi}{z}$. But by trigonometry $z^2 + a^2 - 2az\cos\varphi = r^2$, therefore taking fluxions $z\dot{z} - a\dot{z}\cos\varphi + az\dot{\varphi}\sin\varphi = 0$, or $\frac{-\dot{\varphi}}{\dot{z}} = \frac{z - a\cos\varphi}{az\sin\varphi}$; whence $\frac{\varphi}{z} = \frac{z - a\cos\varphi}{az\sin\varphi}$, or $a\varphi\sin\varphi = z - a\cos\varphi$: But $z = a\cos\varphi + \sqrt{(r^2 - a^2\sin^2\varphi)}$, and $z - a\cos\varphi = \sqrt{(r^2 - a^2\sin^2\varphi)}$; therefore

$$a\varphi\sin\varphi = \sqrt{(r^2 - a^2\sin^2\varphi)},$$

or $(\varphi^2 + 1)\sin^2\varphi = \frac{r^2}{a^2}$. From which equation φ may be easily found by trial and error, and then the radius of the circle becomes known.

When $a = r$, or the given point is in the circumference of the given circle, $(\varphi^2+1)\sin^2\varphi = 1$, or $\varphi^2 = \frac{1}{\sin^2\varphi} - 1 = \text{co-tan}^2\varphi$, or $\varphi = \text{co tan}\,\varphi$, so that in this case we have only to find an arc φ which shall be equal to its co-tangent.

XV. QUESTION 325, *by Mr.* LOWRY.

If two weights be suspended at the ends of two strings which pass over two fixed pullies, and the other ends of the strings be fastened to the extremities of a given beam or rod which is moveable about a fixed point; what will be the velocities of the weights when they move in a vertical direction, the weight of the beam or rod being considered in estimating the motion of the bodies?

SOLUTION, *by Mr.* LOWRY.

Let AB (fig. 133. pl. 4.) be the given beam or rod moveable about the fixed point C; D and E the pullies, and P and Q the given weights, which descend by the force of gravity and draw up the end A of the beam. Through C draw HCI parallel to the horizon, and join CD and CE. Let P*p* and Q*q* be the spaces descended by the weights P and Q in the instant of time $\dot t$, and A*a* and B*b* the spaces described by the ends of the beam in the same time. Draw *am* and *bn* perpendicular to AD and BE respectively; then A*m* is evidently = P*p*, and B*n* = Q*q*. Put a = AC, b = BC, c = CD, d = CE, x = DP, y = EQ, $\dot x$ = P*p* = A*m*, $\dot y$ = Q*q* = B*n*, α = the angle DCH, β = the angle ECI, φ = the angle HCA = BCI, radius 1, e = the distance of the centre of gravity G of the beam from C, and $g = 32\frac{1}{6}$ feet the measure of the accelerative force of gravity.

Then the velocity of P is $= \frac{\dot x}{\dot t}$, and the variation of that velocity, or the measure of the accelerative force, $= \frac{\ddot x}{\dot t^2}$; but the accelerative force of gravity is $= g$, therefore $P \times \left(g - \frac{\ddot x}{\dot t^2}\right)$ is the motive force lost by P in consequence of its connection with the beam, and which is equivalent to the motive force $P \times \left(g - \frac{\ddot x}{\dot t^2}\right) \times \sin Aam$ acting perpendicularly to the beam at A, $= P \times \left(g - \frac{\ddot x}{\dot t^2}\right) \times \frac{Am}{Aa} = P \times \left(g - \frac{\ddot x}{\dot t^2}\right) \times \frac{\dot x}{a\dot\varphi}$. In the same way it appears that the motive force lost by Q is $= \text{Q} \times \left(g - \frac{\ddot{y}}{\dot{t}^2}\right)$, which is equivalent to $\text{Q} \times \left(g - \frac{\ddot{y}}{\dot{t}^2}\right) \times \sin \text{B}bn = \text{Q} \times \left(g - \frac{\ddot{y}}{\dot{t}^2}\right) \times \frac{bn}{\text{B}b} = \text{Q} \times \left(g - \frac{\ddot{y}}{\dot{t}^2}\right) \times \frac{\dot{y}}{b\dot{\varphi}}$ acting perpendicularly to the beam at B: But, by the property of the lever, this force is equivalent to $\frac{b}{a} \times \text{Q} \times \left(g - \frac{\ddot{y}}{\dot{t}^2}\right) \times \frac{\dot{y}}{b\dot{\varphi}}$ applied at A, therefore the whole motive force at A is $=$

$$\text{P} \times \left(g - \frac{\ddot{x}}{\dot{t}^2}\right) \times \frac{\dot{x}}{a\dot{\varphi}} + \text{Q} \times \left(g - \frac{\ddot{y}}{\dot{t}^2}\right) \times \frac{\dot{y}}{a\dot{\varphi}}.$$

Now, by the principles of rotatory motion, $\frac{\text{M}}{a^2} \times \frac{(\text{A}a)^{\cdot\cdot}}{\dot{t}^2} = \frac{\text{M}}{a} \times \frac{\ddot{\varphi}}{\dot{t}^2}$, is the measure of the motive force at A, M being the moment of inertia of the beam, or the sum of the products of each particle by the square of its distance from the axis of motion, and the motive force arising from gravity is $= \frac{e}{a} \times g\,\text{W} \cos \varphi$, W being the weight of the beam; therefore the whole motive force gained by the beam at A is equal to $\frac{\text{M}}{a} \times \frac{\ddot{\varphi}}{\dot{t}^2} + \frac{e}{a} g\text{W} \cos \varphi$, and which, by the principles of motion, must be equal to the motive force lost by the descending weights; therefore

$$\frac{\text{M}}{a} \times \frac{\ddot{\varphi}}{\dot{t}^2} + \frac{e}{a} g\text{W} \cos \varphi = \text{P}\left(g - \frac{\ddot{x}}{\dot{t}^2}\right) \times \frac{\dot{x}}{a\dot{\varphi}} + \text{Q}\left(g - \frac{\ddot{y}}{\dot{t}^2}\right) \times \frac{\dot{y}}{a\dot{\varphi}},$$

and if the whole be multiplied by $2a\dot{\varphi}$ and the fluents taken we get

$$\text{P}\frac{\dot{x}^2}{\dot{t}^2} + \text{Q}\frac{\dot{y}^2}{\dot{t}^2} + \frac{\text{M}\dot{\varphi}^2}{\dot{t}^2} = 2g\left(\text{P}x + \text{Q}y - e\,\text{W} \sin\varphi + \text{C}\right),$$

or $\text{P}v^2 + \text{Q}u^2 + \frac{\text{M}}{a^2}w^2 = 2g\left(\text{P}x + \text{Q}y - e\,\text{W} \sin\varphi + \text{C}\right)$,

v being $= \frac{\dot{x}}{\dot{t}}$, $u = \frac{\dot{y}}{\dot{t}}$ and $w = \frac{a\dot{\varphi}}{\dot{t}}$, the respective velocities of P, Q and A.

Now since $v = \frac{\dot{x}}{\dot{t}}$ and $w = \frac{a\dot{\varphi}}{\dot{t}}$, w is $= v \times \frac{\dot{x}}{a\dot{\varphi}}$, but $\frac{\dot{x}}{a\dot{\varphi}} = \frac{\text{A}m}{\text{A}a} = \sin \text{A}an = \sin \text{CAD} = \frac{\text{DC}}{\text{DA}} \sin \text{ACD} = \frac{d}{l - x} \times \sin$

$(\alpha + \varphi)$, l being put for the length of the string AP; therefore w is $= v \times \frac{d}{l - x} \times \sin(\alpha + \varphi)$. Also since $v = \frac{\dot{x}}{t}$ and $u = \frac{\dot{y}}{t}$, u is $= v \times \frac{\dot{x}}{\dot{y}}$; but $\dot{x} = \mathrm{A}a \sin \mathrm{A}am = \mathrm{A}a \sin \mathrm{CAD} = \mathrm{A}a \times \frac{d}{l - x} \times \sin(\alpha + \varphi)$, and $\dot{y} = \mathrm{B}b \sin \mathrm{B}bn = \mathrm{B}b \sin \mathrm{CBE} = \mathrm{B}b \times \frac{c}{l' - x} \sin(\beta + \varphi) = \mathrm{A}a \times \frac{bc}{a(l' - x)} \sin(\beta + \varphi)$, l' being put for the length of the string BQ; therefore $u = \frac{\dot{x}}{\dot{y}} = v \times \frac{ad \sin(\alpha + \varphi)(l' - x)}{bc \sin(\beta + \varphi)(l - x)}$, and these values of u and w being substituted in the expression $\mathrm{P}v^2 + \mathrm{Q}u^2 + \frac{\mathrm{M}}{a^2}w^2$, and the whole reduced to a common denominator, it becomes

$$v^2 \times \frac{\mathrm{P}b^2c^2\sin^2(\beta+\varphi)(l-x)^2 + \mathrm{Q}a^2d^2\sin^2(\alpha+\varphi)(l'-x)^2 + \frac{\mathrm{M}}{a^2}b^2c^2\sin^2(\alpha+\varphi)\sin^2(\beta+\varphi)}{b^2c^2\sin^2(\beta+\varphi)(l-x)^2}$$

and if R^2 be put for the quantity in the numerator we have

$$\frac{v^2\mathrm{R}^2}{b^2c^2\sin^2(\beta + \varphi)(l - x)^2} = 2g(\mathrm{P}x + \mathrm{Q}y - e\mathrm{w}\sin\varphi + \mathrm{C}),$$

therefore

$$v = \frac{bc\sin(\beta + \varphi)(l - x)}{\mathrm{R}} \times \sqrt{\left\{2g(\mathrm{P}x + \mathrm{Q}y - e\mathrm{w}\sin\varphi + \mathrm{C})\right\}}$$

$$\text{and } u = \frac{ad\sin(\alpha + \varphi)(l' - x)}{\mathrm{R}} \times \sqrt{\left\{2g(\mathrm{P}x + \mathrm{Q}y - e\mathrm{w}\sin\varphi + \mathrm{C})\right\}}$$

the velocities required.

$$\text{Also } w = \frac{bcd\sin(\beta + \varphi)\sin(\alpha + \varphi)}{\mathrm{R}} \times \sqrt{\left\{2g(\mathrm{P}x + \mathrm{Q}y - e\mathrm{w}\sin\varphi + \mathrm{C})\right\}}$$

$$\text{and } w' = \frac{acd\sin(\beta + \varphi)\sin(\alpha + \varphi)}{\mathrm{R}} \times \sqrt{\left\{2g(\mathrm{P}x + \mathrm{Q}y - e\mathrm{w}\sin\varphi + \mathrm{C})\right\}}$$

the velocities of the ends of the beam.

The problem may be resolved in a similar manner when the weights are constrained to move upon given curves, instead of descending in the direction of gravity, provided those curves are in a vertical plane passing through the point about which the beam revolves. For let MN and RS (fig. 134. pl. 4) be the given curves upon which the bodies P and Q descend, and let $\mathrm{P}p = \dot{r}$, and $\mathrm{Q}q = \dot{s}$, be the elementary spaces described by the bodies in the instant of time $\dot{t}$, and $\dot{x}$ and $\dot{y}$ the spaces descended perpendicularly in the same time. Draw Pr and Qs perpendicular to Dp and Eq, and let the other lines be drawn as before. Put $\dot z = pr = $ Am, and let the substitution made in the preceding case be retained; then the velocity of P is $= \frac{\dot r}{\dot t}$, and the accelerative force $= \frac{\ddot r}{\dot t^2}$; but the accelerative force of gravity is now only $g \times \frac{\dot x}{\dot r}$, therefore the motive force lost by P is $=$ P $(g\frac{\dot x}{\dot r} - \frac{\ddot r}{\dot t^2})$, which is equivalent to the motive force P $(g\frac{\dot x}{\dot r} - \frac{\ddot r}{\dot t^2}) \times \frac{\dot r}{\dot z}$ acting in the direction of the string, or to P$(g\frac{\dot x}{\dot r} - \frac{\ddot r}{\dot t^2}) \times \frac{\dot r}{\dot z}\sin$ A$am =$ P $(g\frac{\dot x}{\dot r} - \frac{\ddot r}{\dot t^2}) \times \frac{\dot r}{a\dot\varphi}$ acting perpendicularly to the beam at A. In like manner the motive force lost by Q is $=$ Q $(g\frac{\dot y}{\dot s} - \frac{\ddot s}{\dot t^2}) \times \frac{\dot s}{b\dot\varphi} =$ the motive force Q $(g\frac{\dot y}{\dot s} - \frac{\ddot s}{\dot t^2}) \times \frac{\dot s}{a\dot\varphi}$ applied perpendicularly at A; therefore the whole motive force lost by the weights P and Q in consequence of their connection with the beam is $=$ P $(g\frac{\dot x}{\dot r} - \frac{\ddot r}{\dot t^2}) \times \frac{\dot r}{a\dot\varphi} +$ Q $(g\frac{\dot y}{\dot s} - \frac{\ddot s}{\dot t^2}) \times \frac{\dot s}{a\dot\varphi}$; and which, as before, must be equal to $\frac{M}{a} \times \frac{\ddot\varphi}{\dot t^2} + \frac{e}{a}gW\cos\varphi$, therefore multiplying by $2a\dot\varphi$ and taking the fluents we get

$$P\frac{\dot r^2}{\dot t^2} + Q\frac{\dot s^2}{\dot t^2} + M\frac{\ddot\varphi}{\dot t^2} = 2g\,(Px + Qy - egW\sin\varphi + C),$$

$$\text{or } Pv^2 + Qu^2 + \frac{M}{a^2}w^2 = 2g\,(Px + Qy - egW\sin\varphi + C).$$

But $u = v \times \frac{\dot s}{\dot r}$, and $w = v \times \frac{\dot r}{a\dot\varphi}$, therefore by substitution and reduction

$$v = a\dot r\dot\varphi\sqrt{\frac{2g\,(Px + Qy - egW\sin\varphi + C)}{Pa^2\dot r^2\dot\varphi^2 + Qa^2\dot s^2\dot\varphi^2 + \frac{M}{a^2}\dot r^4}},$$

$$u = a\dot s\dot\varphi\sqrt{\frac{2g\,(Px + Qy - egW\sin\varphi + G)}{Pa^2\dot r^2\dot\varphi^2 + Qa^2\dot s^2\dot\varphi^2 + \frac{M}{a^2}\dot r^4}},$$

where the relation of $\dot{r}$, $\dot{s}$, and $\dot{\varphi}$ must be determined from the nature of the curves, and considering that

$$\mathrm{A}a \times \sin \mathrm{A}am = \mathrm{A}m = pr = \mathrm{P}p \times \sin p\mathrm{P}r$$

$$\text{or } a\dot{\varphi} = r \times \frac{\sin p\mathrm{P}r}{\sin \mathrm{A}am}.$$

By comparing what has been done above, with art. v. part ii. vol. 3 of the Repository, it will be obvious how the problem may be resolved when there are any number of weights attached to the beam, and moveable upon given curves. The problem might also be greatly generalized by supposing the weights to move upon given surfaces, instead of being always in the same vertical plane. The beam would then have a compound rotatory motion, and we might determine not only the velocities of the weights, but also the nature of the curves described on the given surfaces, and of those described by the ends of the beam.

XVI. QUESTION 326, *by Mr.* LOWRY.

Suppose the vertex of a given cone to rest on an inclined plane while the axis is in a vertical direction; what will be the velocity of the centre of gravity of the cone when it is suffered to descend by the force of gravity, the vertex being at liberty to slide freely along the inclined plane without any obstruction whatever?

SOLUTION *by Mr.* LOWRY.

Let AD (fig. 135. pl. 4.) represent the inclined plane, AB an horizontal plane passing through any point A, G the centre of gravity of the cone and C its vertex; and let O be the point on the plane at which the motion of C commences. Then while the point C descends in the direction DA, the centre of gravity G will move in a vertical plane passing through that line, and the body will have a progressive motion in the direction DA, and a rotatory motion about G.

Draw OZ and GH perpendicular to AD, and GE perpendicular to AB, and put $a =$ CG, $\alpha =$ the angle BAD $=$ HGE, $x =$ OH $=$ GZ, $y =$ GH, $\varphi =$ the angle CGH, R $=$ the re-action of the plane at the point C in the direction CF perpendicular to AD, W $=$ the mass of the body, $g = 32\frac{1}{6}$ feet the accelerative force of gravity, $v = \frac{\dot{x}}{\dot{t}}$ the velocity of G parallel to AD, $u = \frac{\dot{y}}{\dot{t}}$ the velocity parallel to OZ, and $w = \frac{\dot{\varphi}}{\dot{t}}$, the angular velocity about G. Now the motion of the centre of gravity is evidently the same as if the whole mass was concentrated at that point, and the motive forces Wg and R applied there in the directions GE and GK. The force of gravity g may be resolved into two other forces, viz. $g \sin \alpha$ acting parallel to AD and tending to augment x; and $g \cos \alpha$ acting in the direction GH and tending to diminish y; therefore the point G is urged in a direction parallel to AD by a motive force $=$ W$g \sin \alpha$, and in the direction GH by the motive force R $-$ W$g \cos \alpha$; therefore by dynamics

$$\frac{\ddot{x}}{\dot{t}^2} = g \sin \alpha \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{W\ddot{y}}{\dot{t}^2} = R - Wg \cos \alpha \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Multiplying the first equation by $2\dot{x}$, and integrating, we have $\frac{\dot{x}^2}{\dot{t}^2} = 2gx \sin \alpha + c$, or $v^2 = 2gx \sin \alpha + c$; but when $v = 0$, $x = a \sin \alpha$, therefore $c = -2ga \sin^2\alpha$, and $v = \sqrt{2g \sin \alpha\,(x - a \sin \alpha)}$, the velocity of G in a direction parallel to the plane.

To find the angular velocity; let M be the moment of inertia of the body with respect to an axis passing through G, at right angles to the vertical plane, or the sum of the products of each particle of the body by the square of its distance from the axis of motion, then R $\sin \varphi$ being the motive force at C in a direction perpendicular to CG, we have by the principles of rotatory motion, $\frac{M\ddot{\varphi}}{\dot{t}^2} = a\text{R} \sin\varphi$, but from equation (2) $\text{R} = \text{W}\left(\frac{\ddot{y}}{\dot{t}^2} + g \cos \alpha\right)$; therefore by substitution $\frac{M\ddot{\varphi}}{\dot{t}^2} = a\text{W}\left(\frac{\ddot{y} \sin \varphi}{\dot{t}^2} + g \cos \alpha \sin \varphi\right)$, and multiplying by $2\dot{\varphi}$,

$$\frac{2M\ddot{\varphi}\dot{\varphi}}{\dot{t}^2} = 2a\text{W}\ddot{y}\dot{\varphi} \sin \varphi + 2a\text{W}g \cos \alpha\dot{\varphi} \sin \varphi,$$

but $a\dot{\varphi} \sin \varphi = -a(\cos \varphi)^{\cdot} = \dot{y}$, therefore

$$\frac{2M\ddot{\varphi}\dot{\varphi}}{\dot{t}^2} + \frac{2\text{W}\ddot{y}\dot{y}}{\dot{t}^2} = -2a\text{W}g \cos \alpha\,(\cos\varphi)^{\cdot}$$

and taking the fluents

$$\frac{M\dot{\varphi}^2}{\dot{t}^2} + \frac{\text{W}\dot{y}^2}{\dot{t}^2} = c' - 2a\text{W}g \cos \alpha \cos \varphi,$$

or $Mw^2 + \text{W}u^2 = c' - 2a\text{W}g \cos \alpha \cos \varphi$.

But since $u = \frac{\dot{y}}{\dot{t}}$ and $w = \frac{\dot{\varphi}}{\dot{t}}$, w^2 is $= u^2 \times \frac{\dot{\varphi}^2}{\dot{y}^2} = u^2 \times \frac{1}{a^2 \sin^2\varphi}$, therefore

$$u^2\left(\frac{\mathrm{M}}{a^2\sin^2\phi}+\mathrm{W}\right)=c'-2a\mathrm{W}g\cos\alpha\cos\phi:$$

whence $u=a\sin\phi\sqrt{\frac{c'-2a\mathrm{W}g\cos\alpha\cos\phi}{\mathrm{M}+\mathrm{W}a^2\sin^2\phi}}$

and $w=\sqrt{\frac{c'-2a\mathrm{W}g\cos\alpha\cos\phi}{\mathrm{M}+\mathrm{W}a^2\sin^2\phi}}$.

It is obvious that this solution is applicable to any solid whatever, provided it slides on a point or apex at C. For the differences of the velocities arise entirely from the different values of a and M. When the body is an indefinitely slender rod having its centre of gravity in the middle, M is $=\mathrm{W}\times\frac{a^2}{3}$, therefore if β be put for the value of ϕ at the commencement of the motion we have

$$u=\sin\phi\sqrt{\frac{6ag\cos\alpha\,(\cos\beta-\cos\phi)}{1+3\sin^2\phi}},$$

and $w=\sqrt{\frac{6ag\cos\alpha\,(\cos\beta-\cos\phi)}{a+3a^2\sin^2\phi}}$.

The motion of the centre of gravity may be determined in a similar manner when the extremities of the body move upon two given planes, or when a rod is connected to the centre of gravity of the body and its ends move on those planes. Suppose, for example, it were required to find the velocity of the centre of gravity G (fig. 136, pl. 4) of the rod CD, or of the body connected to that rod, when the end C moves on the horizontal plane OA, and the other end D descends on the vertical plane BO. Draw GH perpendicular to OA, and put $x=$ OH, $y=$ GH, $a=$ CG, $c=$ GD, $\phi=$ the angle CGH, $v=\frac{\dot{x}}{\dot{t}}$ the velocity parallel to AO, $u=\frac{\dot{y}}{\dot{t}}$ the velocity parallel to DO, $w=\frac{\dot{\phi}}{\dot{t}}$ the angular velocity about G, R $=$ the re-action of the plane at C in the direction CF perpendicular to AO, and R$'$ $=$ the re-action of the plane at D in the direction DE perpendicular to OB, then we have as before

$$\mathrm{W}\frac{\ddot{x}}{\dot{t}^2}=\mathrm{R}',$$

$$\mathrm{W}\frac{\ddot{y}}{\dot{t}^2}=-g\mathrm{W}+\mathrm{R},$$

$$\mathrm{M}\frac{\ddot{\phi}}{\dot{t}^2}=a\mathrm{R}\sin\phi-c\mathrm{R}'\cos\phi;$$

and if the values of R$'$ and R be found from the first and second equations, and substituted in the third, we have

$$\text{M}\frac{\ddot{\varphi}}{\dot{t}^2} = \text{W}\left(\frac{a\ddot{y}}{\dot{t}^2}\sin\varphi - \frac{c\ddot{r}}{\dot{t}^2}\cos\varphi + ag\sin\varphi\right);$$

therefore multiplying by $2\dot{\varphi}$ and taking the fluents, remarking at the same time that $a\dot{\varphi}\sin\varphi = -a(\cos\varphi)^{\cdot} = -\dot{y}$ and $c\dot{\varphi}\cos\varphi = (c\sin\varphi)^{\cdot} = \dot{x}$, we get

$$\frac{\text{M}\dot{\varphi}^2}{\dot{t}^2} = \text{W}\left(-\frac{\dot{y}^2}{\dot{t}^2} - \frac{\dot{x}^2}{\dot{t}^2} + 2g(d-y)\right),$$ d being the value of y at the commencement of the motion,

$$\text{or } \text{M}w^2 + \text{W}u^2 + \text{W}v^2 = 2\text{W}g(d-y).$$

$$\text{But } u^2 = \frac{v^2\dot{y}^2}{\dot{x}^2} = \frac{a^2\sin^2\varphi}{c^2\cos^2\varphi} = \frac{a^4x^2}{c^4y^2}$$

$$\text{and } w^2 = \frac{v^2\dot{\varphi}^2}{\dot{x}^2} = \frac{1}{c^2\cos^2\varphi} = \frac{a^2}{c^2y^2},$$

therefore $v^2\left(\frac{\text{M}}{c^2\cos^2\varphi} + \frac{\text{W}a^2\sin^2\varphi}{c^2\cos^2\varphi} + \text{W}\right) = 2\text{W}g(d-y)$,

$$v = c\cos\varphi\sqrt{\frac{2g(d-y)}{\text{M} + \text{W}a^2\sin^2\varphi + \text{W}c^2\cos^2\varphi}},$$

$$u = a\sin\varphi\sqrt{\frac{2g(d-y)}{\text{M} + \text{W}a^2\sin^2\varphi + \text{W}c^2\cos^2\varphi}},$$

$$\text{and } w = \sqrt{\frac{2g(d-y)}{\text{M} + \text{W}a^2\sin^2\varphi + \text{W}c^2\cos^2\varphi}}.$$

When the rod is homogeneous, and indefinitely slender, the centre of gravity is in the middle, or $a = c$; and M is then $= \text{W} \times \frac{a^2}{3}$, and the above expressions for the velocities become

$$v = \frac{y}{2a}\sqrt{6g(d-y)}$$

$$u = \frac{x}{2a}\sqrt{6g(d-y)}$$

$$\text{and } w = \frac{1}{2a}\sqrt{6g(d-y)}$$

agreeing with those determined by Mr. Landen, in the Ladies' Diary for 1758, by a different method.

When the given planes are both inclined to the horizon the calculation is rather more tedious, but the method of proceeding is precisely the same. For draw GH (fig. 137. pl. 4.) perpendicular to OA, and GK perpendicular to the horizontal plane AX. Put $x = $ OH, $y = $ HG, $\alpha = $ the angle OAX $=$ HGK, $\beta = $ the angle BOS, $\varphi = $ the angle CGH; then the force R′, in the direction DE, perpendicular to the plane OB, may be resolved into two others, viz. R′ $\sin\beta$, acting parallel to OA, and R′ $\cos\beta$, acting perpendicularly to that line. The force of gravity may also be resolved into two others $g\sin\alpha$ and $g\cos\alpha$, the former parallel to AO, and the latter perpendicular thereto; therefore the point G is urged in a direction parallel to OA by the motive force $Wg\sin\alpha + R'\sin\beta$, and in the direction GH by the motive force $-Wg\cos\alpha + R'\cos\beta + R$. But the motive force at C, in a direction perpendicular to the rod, is $= R\sin\phi$, and that at D in the same direction $= R'\sin(\phi + \beta)$; therefore we have the three equations

$$W\frac{\ddot{x}}{t^2} = Wg\sin\alpha + R'\sin\beta,$$

$$W\frac{\ddot{y}}{t^2} = -Wg\cos\alpha + R'\cos\beta + R,$$

$$M\frac{\ddot{\phi}}{t^2} = aR\sin\phi - cR'\sin(\phi + \beta),$$

and by exterminating R and R', and proceeding in all respects as before, we shall obtain the values of v, u and w in terms of ϕ and known quantities.

XVII. QUESTION 327, *by Mr.* W. WALLACE, *R. M. College.*

Let there be three curves of such a nature, that if a tangent be drawn to one of them, a given portion of that tangent is intercepted between the other two: then, normals to the three curves at the points of intersection and contact shall pass through the same point. Required the demonstration?

SOLUTION, *by Mr.* LOWRY.

Let AB (fig. 138. pl. 4.) be the given portion of the tangent intercepted between the given curves, and O the point of contact. Let AX and BY be tangents, and AP and BP normals to the curves at A and B; then it is to be proved that PO is perpendicular to AB, or a normal to the curve at O. Now let ab be another position of the tangent indefinitely near to AB and meeting it in o, and draw Am and Bn perpendicular to ab; then because ab is $=$ AB by hypothesis, am is $= bn$; and in the little elementary triangles Aam, Bbn, we have Am $= am$ tan Aam, and Bn $= bn$ tan Bbn; therefore

Am : Bn :: tan Aam : tan Bbn,
but Am : Bn :: oA : oB, therefore
oA : oB :: tan Aam : tan Bbn.

But since ab is indefinitely near to AB, the point o is indefinitely near to O, and consequently the lines Ao, and Bo approach indefinitely near to an equality with AO and BO, and the angles

A*am* and B*bn* with OAX and OBY ; the one may therefore be substituted for the other respectively, and the above proportion will then become

OA : OB :: tan OAX : tan OBY,

or because OAX is the complement of OAP and OBY of OBP,

OA : OB :: cot OAP : cot OBP ;

therefore PO is perpendicular to AB.

The proposition is also true, and may be demonstrated in a similar manner, when the curves are upon the surface of a sphere.

Cor. If a straight line AB of a given length, continue to touch a given curve while one of its ends, as A, is moved upon a given line of any kind, and normals be drawn at A and the point of contact O to meet in P, the line PB will be a normal to the curve which is the locus of B.

XVIII. QUESTION 328, *by Mr.* W. WALLACE.

Let A,B, two angles of a triangle ABC given in magnitude be upon lines of any kind. Draw AP, BP, normals to these lines at A and B, and join CP; this last line shall be a normal to the line that is the *locus* of C the remaining angle of the triangle. Required the demonstration?

SOLUTION, *by Mr.* LOWRY.

Let *abc* (fig. 139. pl. 4.) be another position of the triangle indefinitely near to ABC and having its angular points *a* and *b* on the given curves; and let the sides *ab* and *ac* intersect AB and AC in O and O′ respectively. Also let AX, BY and CZ be tangents to the curves at A, B, and C. Draw A*m* and B*n* perpendicular to *ab* ; then because *ab* = AB, *am* is = *bn* ; therefore in the triangles A*am*, B*bn*,

A*m* : B*n* (:: OA : OB) :: tan A*am* : tan B*bn*,

but ultimately the angle A*am* is = OAX and the angle B*bn* = OBY ; therefore

OA : OB :: tan OAX : tan OBY :: cot OAP : cot OBP ;

therefore OP is perpendicular to AB.

Now join OO′, then because the angle O*a*O′ is = the angle OAO′, the points O, *a*, A, O′ are in a circle; therefore the angle AO′O is = A*am* = *a*AO = ABO, wherefore the points A, O′, P, O are in a circle; therefore the angle AO′P is a right angle. Draw A*r* and C*s* perpendicular to *ca*, then because AC = *ac*, *ar* is = *cs*, therefore in the elementary triangles A*ra*, C*rc*,

A*r* : C*s* :: tan A*ar* : tan C*cs*,

but ultimately the angle Aar = O′AX = the comp. of O′AP and angle Ccs = O′CZ; also Ar : Cs :: O′A : O′C,

therefore O′A : O′C :: cot O′AP : tan O′CZ:

but by trigonometry the segments O′A, O′C made by the perpendicular PO′ are as the cotangents of the adjacent angles O′AP, O′CP, therefore the tangent of O′CZ is equal to the cotangent of PCO; therefore the one angle is the complement of the other, and consequently CP is perpendicular to CZ or a normal to the curve at C.

XIX. QUESTION 329, *by Mr.* W. WALLACE.

Three curves being given by position, shew how a tangent may be drawn to one of them so that a given portion of that tangent may be intercepted between the other two.

SOLUTION, *by Mr.* LOWRY.

Let CHV, BGW, and XPY (fig. 140. pl. 4.) be the given curves, referred to the line CB as an axis, and HG the given tangent touching the curve XY in P, and terminating in the other curves at H and G. Produce HG to meet the axis in K, and draw the ordinates PE, GD and HF perpendicular to CB; then supposing the abscissas to be reckoned from a given point A, put AE $= x$, AD $= x'$, AF $= x''$, EP $= y$, DG $= y'$, HF $= y''$, HG $= c$, and let $\dot{x} = p\dot{y}$ be the fluxional equation of the curve XPY, p being a given function of x and y; then the subtangent EK is $= py$, and DK $=$ EK $-$ EA $-$ AD $= py - x - x'$; but by similar triangles EK : EP :: DK : DG, or $py : y :: py - x - x' : y'$; therefore $y - \frac{x' + x}{p} = y'$. In like manner FK $=$ KE $+$ AF $-$ AE $= py + x'' - x$, and by similar triangles $py : y :: py + x'' - x : y''$, therefore $y + \frac{x'' - x}{p} = y''$. Again PK $= \sqrt{(\text{PE}^2 + \text{EK}^2)} = y\sqrt{(1 + p^2)}$, and by similar triangles KE : PK :: FD : HG, that is, $py : y\sqrt{(1 + p^2)} :: x' + x'' : c$, therefore $x'' + x' = \frac{cp}{\sqrt{(1+p^2)}}$. But since the nature of the curves are known, y' is a given function of $x' = f(x')$, and y'' a given function of $x'' = f'(x'')$; therefore we have the three equations

$$y - \frac{x' + x}{p} = f(x') \ldots\ldots\ldots\ldots\ldots (1)$$

$$y + \frac{x'' - x}{p} = f'(x'') \ldots\ldots\ldots\ldots (2)$$

$$x' + x'' = \frac{cp}{\sqrt{(1+p^2)}} \ldots\ldots\ldots\ldots (3)$$

and from these, by exterminating x' and x'', we shall get an equation with only x, y and p; and since any two of these quantities, may be expressed in terms of the third, by means of the equation of the curve, we shall finally have an equation containing only one of them and known quantities; and then the position of PE will be determined and the problem completely resolved.

Instead of exterminating x' and x'', it will sometimes be more convenient to retain one of those quantities, and exterminate x and its functions y and p: but the expediency of this depends entirely on the nature of the functions $f(x')$ and $f'(x'')$.

When CHV and BGW are straight lines given by position, the functions $f(x')$ and $f'(x'')$ are of the simplest nature possible, and we immediately obtain a general formula for resolving the problem whatever may be the nature of the curve XY. For if α and β be put for the tangents of the angles DBG and VCF, and AB $= a$ and AC $= b$, then BD $= a - x'$, and CF $= b - x''$; therefore $f(x') = \alpha(a - x')$, and $f'(x'') = \beta(b - x'')$; and equations (1), (2) then become

$$y - \frac{x' + x}{p} = \alpha(a - x')$$

$$y + \frac{x'' - x}{p} = \beta(b - x'');$$

from the first of these $x' = \dfrac{py - x - ap\alpha}{1 - p\alpha}$,

and from the second $x'' = \dfrac{x - py + pb\beta}{1 + p\beta}$;

therefore by equation 3),

$\dfrac{py - x - ap\alpha}{1 - p\alpha} + \dfrac{x - py + pb\beta}{1 + p\beta} = \dfrac{cp}{\sqrt{(1 + p^2)}}$, and by reduction

$$\left.\begin{matrix} \{b\beta - a\alpha + (py - x)(\alpha + \beta) - (a + b)p\alpha\beta\}\sqrt{(1 + p^2)} \\ - c \times (1 + p(\beta - \alpha) - p^2\alpha\beta) \end{matrix}\right\} = 0,$$

in which equation we have only to substitute for two of the unknown quantities their values in terms of the other, and then there will arise an equation with only one unknown quantity.

Suppose, for example, that the curve XPY is a parabola whose vertex is A, and parameter $= e$; then $ex = y^2$, $e\dot{x} = 2y\dot{y}$, $p = \dfrac{2y}{e}$ and $\sqrt{(1 + p^2)} = \dfrac{\sqrt{(e^2 + 4y^2)}}{e}$, therefore, by substitution,

$$\left\{\begin{matrix} \Big((eb\beta - ea\alpha) + y^2(\alpha + \beta) - 2(a + b)\alpha\beta\Big)\sqrt{(e^2 - 4y^2)} \\ - c \times \Big(e^2 + 2ey(\beta - \alpha) - 4y^2\alpha\beta\Big) \end{matrix}\right\} = 0,$$

which being squared produces an equation of the sixth power.

But in the case of two straight lines and a curve, the result will in most cases be simpler if the curve is referred to one of the lines as an axis, and the abscissa reckoned from the point where they intersect. Thus if VB and VC (fig. 141. pl. 4.) be the lines, and XPY the curve, PK parallel to VC, VK $= x$, PK $= y$, and angle CVB $= e$; then KG $= py$, as before, and VG $= x + py$, GP$^2 = y^2p^2 + y^2 - 2py^2 \cos e = y^2 \times (p^2 + 1 - 2p \cos e)$, and by similar triangles GK^2 : GP^2 :: GV^2 : GH^2, or $y^2p^2 : y^2 \times (p^2 + 1 - 2p \cos e) :: (x + py)^2 : c^2$; therefore $(x + py)^2(p^2 + 1 - 2p \cos e) - p^2 c^2 = 0$, the general formula in this case.

It is obvious that the unknown quantities may be so involved in the general equations (1), (2), (3), as to render their separation next to impossible; in this case we may proceed as follows, and indeed in most cases this will be the best way of resolving the problem: Take as many points P, P′, P″, &c. in the given curve XPY as may be necessary, and as near to one another as possible, and through these points draw tangents, PG, P′G′, P″G″, &c. to terminate in the curve BW at G, G′, G″, &c. Upon these tangents set off GH, G′H′, G″H″, &c. each equal to the given length; through the points H, H′, H″, &c. trace the curve H H′ H″, &c. and where it intersects the given curve CV as at H, will be the point from whence the tangent must be drawn.

The equation of the curve H H′ H″, &c. which is the locus of the extremities of the tangents, may be found from the equations $y - \frac{x' + x}{p} = f(x')$, $y + \frac{x'' - x}{p} = y''$, and $x' + x'' = \frac{cp}{\sqrt{(1 + p^2)}}$ by exterminating x (and its functions y, p) and x', for then we shall have an equation containing only x'' and y'', and known quantities.

There is another way in which the problem might be resolved, but it does not appear to be so convenient as the methods which have been already explained. If the given line GH be supposed to move so as to have its extremities on the given curves BW and CV, we can always find (as in quest. 279, and 307) the curve to which this line is a tangent, and then it would only be required to draw GH to touch this curve and the given one XY; but there would, in general, be as much trouble in finding the subsidiary curve as in resolving the problem, and even then we should still have to draw a tangent to the two curves.

XX. PRIZE QUESTION 400, *by Mr.* LOWRY.

A Dog and a Fox start at the same instant from two places M and N, and run at the uniform rate of *m* and *n* yards in a minute respectively: the dog proceeds in a straight direction MS, but the wily fox pursues such a course as prevents him from being seen by the dog, a fixed object on the level plain being always in a direct line between them. Required the nature of the curve described by the fox, when the distances of the object from M and and N, and the angle NMS are given?

FIRST SOLUTION, *by Mr.* LOWRY, *the Proposer.*

Let O (fig. 142. pl. 4.) be the object, and NFX the curve described by the fox. Through O draw BOA perpendicular to MS, and any other line FD to meet SM in D, and the curve in F: then D and F will be cotemporary positions of the dog and fox, and by the question MD : NF :: $m : n$. Draw FE perpendicular to OB, and put OA $= a$, AM $= b$, OE $= x$, FE $= y$, are NF $= z$ and MD $= w$, then $w : z :: m : n$; therefore $\dot{w} : \dot{z} :: m : n$, or $m\dot{z} = n\dot{w}$. But $x : y :: a :$ AD $= \frac{ay}{x}$; therefore $w = b - \frac{ay}{x}$, and $^{\cdot} = \frac{-a\dot{y}x + ay\dot{x}}{x^2}$: but $\dot{z} = \surd(\dot{x}^2 + \dot{y}^2)$, therefore $m\surd(\dot{x}^2 + \dot{y}^2) = na\left(\frac{y\dot{x} - x\dot{y}}{x^2}\right)$.

To integrate this equation, put $\dot{y} = u\dot{x}$; then it becomes $mx^2 \surd(1 + u^2) = nay - naxu$; and taking fluxions, $2mx\dot{x}\surd(1 + u^2) + \frac{mx^2 u\dot{u}}{\surd(1 + u^2)} = na\dot{y} - nax\dot{u} - na\dot{x}u$, or putting $u\dot{x}$ for $\dot{y}$ and dividing the whole by $2mx \times (1 + u^2)^{\frac{1}{4}}$,

$$\dot{x} \times (1 + u^2)^{\frac{1}{4}} + x \times \frac{u\dot{u}}{2(1+u^2)^{\frac{3}{4}}} = \frac{na}{2m} \times \frac{-\dot{u}}{(1 + u^2)^{\frac{1}{4}}};$$

therefore taking the fluents

$$x \times (1 + u^2)^{\frac{1}{4}} = \int -\frac{na}{2m} \times \frac{\dot{u}}{(1 + u^2)^{\frac{1}{4}}}.$$

Let AEG (fig. 143. pl. 4.) be an equilateral hyperbola whose centre is S and semi-axis AS $= 1$, and MSA an angle whose tangent AM is $= u$, then if SE be drawn to bisect the angle MSA, and meet the curve in E, the arc AE is $= \int \frac{\dot{u}}{2(1 + u^2)^{\frac{1}{4}}}$ (See art. 801. Mac Laurin's Fluxions) therefore if A be put = this arc, and C = the proper correction of the fluents (obtained by finding the value of u in the equation $mx^2 \surd(1 + u^2) = nay - naxu$, when for x and y their known values at the commencement of the motion are substituted,) we shall have

$$x \times (1 + u^2)^{\frac{1}{4}} = \frac{na}{m}(\text{C} - \text{A});$$

therefore $x = \frac{na\ (\mathrm{C} - \mathrm{A})}{m(1 + u^2)^{\frac{1}{4}}}$, and $y = \frac{na}{m}\left(\left(\mathrm{C} - \mathrm{A}\right)^2 + \frac{u(\mathrm{C}-\mathrm{A})}{(1+u^2)^{\frac{1}{2}}}\right)$

And from these values of the co-ordinates the curve may be readily constructed.

SECOND SOLUTION *by Mr.* WALLACE.

Let O (fig. 144. pl. 4.) be the fixed object, MON any indefinite position of the line joining the dog and fox, and AOB that position which is perpendicular to MS, the path of the dog: draw ND perpendicular to AO. Put AO $= a$, OD $= x$, DN $= y$, arc BN $= z$, tangent of angle BON $= t$, then MA $= at$.

By the nature of the figure, $1 : t :: x : y$; and by the question, $m : n :: at : z$; hence we have these two equations,

$tx = y$........(1). $nat = mz$........(2).

Let φ be such an angle that $dx = \cos\varphi\, dz$; then $dy = \sin\varphi\, dz$. From equations (1) and (2) we get

$tdx + xdt = dy$,.............. .... $nadt = mdz$;

by substituting for dx, dy, and dt, and dividing by dz, we find

$na\ (\sin\varphi - t\cos\varphi) = mx$........(3)

and taking the fluxions, and putting $\cos\varphi\, dz$, or its equal $na \cos\varphi\, dt$, for mdx, we get

$2\cos\varphi\, dt - t\sin\varphi\, d\varphi = \cos\varphi\, d\varphi$;

Each side of this equation becomes a complete fluxion if divided by $2\sqrt{(\cos\varphi)}$, for it is then $\sqrt{(\cos\varphi)}\, dt - \frac{\frac{1}{2}\sin\varphi}{\sqrt{(\cos\varphi)}}\, td\varphi = \frac{1}{2}\sqrt{(\cos\varphi)}\, d\varphi$,

or $d(\sqrt{(\cos\varphi)}\, t) = \frac{1}{2}\sqrt{(\cos\varphi)}\, d\varphi$..................(4).

In an equilateral hyperbola whose semiaxes $= 1$, let a line be drawn from the centre making with the axis an angle $= \frac{1}{2}\varphi$, and let a tangent be drawn at the point in which it meets the curve, and a perpendicular from the centre to the tangent, then putting Δ for the difference between the hyperbolic arc and the portion of the tangent between the point of contact and perpendicular, it is known that $\frac{1}{2}\sqrt{(\cos\varphi)}\, d\varphi = d\Delta$; therefore equation (4) gives us $d(\sqrt{(\cos\varphi)}\, t) = d\Delta$, and hence, integrating, putting c for the constant correction, and dividing by $\sqrt{(\cos\varphi)}$, we get

$$t = \frac{\Delta + c}{\sqrt{(\cos\varphi)}}.$$

The value of c may be found by putting for φ its value deduced from the equation $na\ (\sin\varphi - t\cos\varphi) = mx$, in the case when x and y, and consequently $t = x \div y$, have known values, as supposed in the question. The curve may be constructed by giving φ all possible values and finding the corresponding values of t, and x, or y.

End of the first Part of the third Volume.

Printed by W. Glendinning, 25, Hatton Garden, London.

Y.

chol-
iples
posi-
mber
gons
d *m*.
rlow,
athe-
his
the
sive,
ount
d in
r last
non-
por-
you
e so
it.

GH.

An

there

Ar
readi

L
positi
whic
pendi
tange
By
tion,

Le
dz.
$tdx +$
by su

and t
$\varphi\, dt$,

Ea

ded b

$\frac{1}{2}\surd$ (
or d(
In
draw
let a t
and a
for t
the ta
know
$d(\surd$
const

Th
ced f
when
as su
givin
of t,

THE

MATHEMATICAL REPOSITORY.

VOL. III. PART II.

ORIGINAL ESSAYS

ARTICLE I.

To the Editor of the Mathematical Repository.

SIR,

I published an Essay on Polygonal Numbers in Nicholson's Journal, vol. XX. p. 161; which contains the principles of a general demonstration to M. de Fermat's singular proposition; viz. Every whole number is either a polygonal number of a given denomination *m*; or it may be divided into polygons of that denomination the number of which does not exceed *m*. This essay was honoured by your correspondent Mr. Barlow, with certain remarks, both critical and logical, but not mathematical. These strictures were specifically answered in his own way, viz. critically and logically: he indeed replied to the defence of the essay; but his observations were inconclusive, because they did not confront my arguments. On this account the gentleman in question may be fairly said to have failed in his refutation of my paper: he, however, maintains in your last number, that the proposition has not received a general demonstration. Now with a view to give my opponent every opportunity to confirm what he has lately advanced, I have sent you a corrected copy of the essay, in which the principles are so applied, than he must easily detect every error contained in it.

Middleshaw, I remain, &c.
Jan. 5, 1810. JOHN GOUGH.

An Essay on Polygonal Numbers. *By Mr.* JOHN GOUGH.

Definitions. 1st. The sum of any arithmetical progression, beginning with unity, is called a polygonal number, or briefly a polygon.

2nd. The difference of two adjacent terms in the generating series increased by 2 is denoted by m, and called the denomination of the polygon; moreover the number is said to be digonal, trigonal, tetragonal, &c. accordingly as $m = 2, 3, 4,$ &c.

3rd. The number of terms a, which are added together to form any polygonal number, is called its index; and the polygon is said to be of the first, second, or third order, &c. accordingly as $a = 1, 2, 3,$ &c.

PROP. I. Put $A = a + t$; then the difference of $A^2 - A$ and $a^2 - a = 2at + t^2 - t$; as is evident by involution and subtraction.

Cor. 1. If $\underset{\cdot}{a} = \frac{a^2 - a}{2}$ we have as $\underset{\cdot}{a} : a :: a - 1 : 2$; hence the ratio of $\underset{\cdot}{a}$ to a increases when a increases.

Cor. 2. When $t = 1$, the above difference is $= 2a$, and half the said difference $= a$.

Cor. 3. Hence it follows, that if we successively put $a = 1, 2, 3,$ &c. and $\frac{a^2 - a}{2} = 0, 1, 3,$ &c. the annexed table may be constructed, by adding unity to each number in the upper line, for the next succeeding value of a in the same; and by adding a to its corresponding function $\frac{a^2 - a}{2}$ for the function of $a + 1$,

$$a = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 : 11$$

$$\frac{a^2 - a}{2} = 0 \cdot 1 \cdot 3 \cdot 6 \cdot 10 \cdot 15 \cdot 21 \cdot 28 \cdot 36 \cdot 45 \cdot 55.$$

Cor. 4. Any positive number n may be assumed for the function of a; for $a^2 - a = 2n$ has a positive root in which $t = 0$, or a fraction, either rational or surd.

PROP. II. Let $\underset{\cdot}{w}$ be a function of the table, whose index $= w$; $\underset{\cdot}{w}$ may be resolved into inferior functions; viz. $\underset{\cdot}{w} = \underset{\cdot}{a} + \underset{\cdot}{b}$, &c. whose indices $= a, b,$ &c. but in all such cases $a + b > w$. For let $\underset{\cdot}{a}$ be the next inferior function to $\underset{\cdot}{w}$; and $\underset{\cdot}{a} + a = w$, by Cor. 2. Prop. I. and $a = w - 1$;

but

but a is a function or the aggregate of functions, the sum of whose indices $(u) > 1$; therefore $a + u = w + u -$ [illegible] $> w$. Again let $\underset{\cdot}{b}$ be the next function to $\underset{\cdot}{a}$, and $\underset{\cdot}{b} + b = \underset{\cdot}{a}$; $b = a - 1$; but b may be resolved into functions, the sum of whose indices $(v) > 1$; therefore $b + v > a$, and $b + v + u > w$.

Cor. 1. It appears, from the demonstration, that the less the sum of the indices is, the fewer are the functions constituting $\underset{\cdot}{w}$.

Cor. 2. What has been proved of w, may be proved, by addition, of the sum of any functions $\underset{\cdot}{w} + \underset{\cdot}{x} + \underset{\cdot}{y}$ &c.

Cor. 3. If $\underset{\cdot}{s}$, $\underset{\cdot}{S}$ be aggregates of functions, the sum of whose indices are s, S; and if $\underset{\cdot}{s} > \underset{\cdot}{S}$ but $s \angle S$; then the functions into which $\underset{\cdot}{s}$ is resolved by s are not more than those into which $\underset{\cdot}{S}$ is resolved by S. For put $\underset{\cdot}{r} = \underset{\cdot}{s} - \underset{\cdot}{S}$; and its index r may be found by Cor. 4. Prop. I. Hence $S + r$ will resolve $\underset{\cdot}{s}$ into more functions than s, by Cor. 1; but $S + r$ resolves $\underset{\cdot}{s}$ into more functions by one than S resolves $\underset{\cdot}{S}$; hence the truth of the Corollary.

PROP. III. If k be a polygon of which the denomination $= m$, and index $= a$; $k = a + (m - 2) \times \frac{a^2 - a}{2}$.

For the first term of the generating series $= 1$ (by Def. 1), but the term of which the number is $a = 1 + (m - 2) \times (a - 1)$ by Def. 2, and Emerson's Proportion, Prop. 6, but $k =$ sum of the terms by Def. 1; hence $k = a + (m - 2) \times \frac{a^2 - a}{2}$ by proportion, Prop. 7.

Cor. Every polygon of the first order $= 1$; and every one of the second order $= m$.

PROP. IV. Let g, h, l, &c. be polygonal numbers of a common denomination m, the indices of which are b, c, d, &c. respectively; also put $b + c + d$, &c. $= a$, and let k be the polygon of which the denomination $= m$, and index $= a$. Then $g + h + l$, &c. $+ (m - 2) \times (bc + bd + cd$, &c.$) = k$.

For by addition and Pr. III. $g + h + l$, &c. $= b + c + d$, &c. $+ (m - 2) \times \frac{b^2 + c^2 + d^2 \text{ \&c.}}{2} - (m - 2 \times \frac{b + c + d \text{ \&c.}}{2}$; but $\frac{b^2 + c^2 + d^2 \text{ \&c.}}{2} = \frac{a^2}{2} - (bc + bd + cd$ &c.$)$ by involution;

hence

hence by substitution $g + h + l$ &c. $= a + (m - 2) \times \frac{a^2 - a}{2} - (m - 2) \times (bc + bd + cd$ &c.); therefore $g + h + l$ &c. $+ (m - 2) \times (bc + bd + cd$ &c.$) = a + (m - 2) \times \frac{a^2 - a}{2} = k$, by Prop. III.

SCHOLIUM. The present proposition generalizies a theorem formerly confined to squares by extending it to all functions of the form $a + (m - 2) \times \frac{a^2 - a}{2}$, including polygons of every denomination, by Prop. III.

PROP. V. Put $e = g + k + l$ &c.; $v = bc + bd + cd$ &c. (in Prop. IV.); then $v = \frac{a^2 - a}{2} \times \frac{a - e}{m - 2}$.

For $e + (m - 2) \times v = k$ (by Prop. IV.) $= a + (m - 2) \times \frac{a^2 - a}{2}$ (by Prop. III.): hence $v = \frac{a^2 - a}{2} \times \frac{a - e}{m - 2}$.

Cor. 1. Since b, c, d, &c, are integers (by hypothesis), v is an integer; but $\frac{a^2 - a}{2}$ is an integer, therefore $\frac{e - a}{m - 2}$ is an integer; hence, if $\frac{e}{m - 2}$ give a quotient s and a remainder p, $a = p$, or $\frac{a}{m - 2}$ gives a quotient r and a remainder p; from which we have the following general expressions, $e = p + (m - 2) \times s$; $a = p + (m - 2) \times r$; $v = \frac{a^2 - a}{2} + r - s$.

Cor. 2. Since we have $p \angle m - 2$, and $e = p + (m - 2) \times s$, where s may be assumed, e is any whole number whatever; consequently every integer is a polygon of the denomination m, or may be resolved into polygons of that denomination, by taking $a = p + (m - 2) \times r$.

SCHOLIUM. I admit that the preceding corollary may be considered as an axiom; but nothing should be assumed in a mathematical inquiry which can be demonstrated.

PROP. VI. $b^2 + c^2 + d^2$ &c. $= a + \frac{2e - 2a}{m - 2} = a + 2s - 2r$.

For

For $b^2 + c^2 + d^2$ &c. $+ 2v = a^2$, and $b^2 + c^2 + d^2$ &c. $= a^2 - 2v = a + \frac{2e - 2a}{m - 2}$ (by Prop. V.) $= a + 2s - 2r$ (Cor. 1. Prop. V.).

Cor. 1. $\frac{b^2 - b + c^2 - c + d^2 - d \text{ \&c.}}{2} = s - r$, because $b + c + d$ &c. $= a$.

Cor. 2. But $\frac{b^2 - b + c^2 - c + d^2 - d \text{ \&c.}}{2}$ = the sum of certain numbers in the second column of the table in Prop. 1; therefore the quantity $\frac{e - a}{m - 2}$, or $s - r$ = the same sum, because $e = g + h + l$ &c. $= a + (m - 2) \times \frac{b^2 - b + c^2 - c + d^2 - d \text{ \&c.}}{2}$ (by Prop. III. and Cor. 2. Prop. V.)

Cor. 3. Now a is any term of an arithmetical progression, bounded by e and p, the common difference of which $= m - 2$; therefore $s - r$ is a similar progression, bounded by 0 when $a = e$, and by s when $a = p$, the common difference of which $= 1$: hence $s - r$ increases as $a = b + c + d$ &c. diminishes; consequently a ultimately becomes too small to resolve $s - r$ into numbers of the table (by Cor. 1. Prop. I.).

PROP. VII. *Problem.* Put $a = b + c + d$ &c.; $v = bc + bd + cd$ &c.; $e = g + h + l$ &c. (as in Prop. IV. and V.): To find an expression for a in terms of e, v, and m?

Here we have $\frac{a^2 - a}{2} + \frac{a - e}{m - 2} = v$ (by Prop. V.); therefore $a^2 - (m - 4) \times \frac{a}{m - 2} = 2v + \frac{e}{m - 2}$. Now put $\frac{m - 4}{m - 2} = p = -1, 0, \frac{1}{3}, \frac{2}{4}$ &c. as $m = 3, 4, 5, 6$, &c.; and we get $a^2 - pa = 2v + \frac{2e}{m - 2}$, which expression always gives a possible and positive value of a.

Examples. 1st. Let c, d, &c. $= 0$; then $b = a$ and $bc + bd + cd$ &c. $= 0$; hence $2v = 0$, and $a^2 - pa = \frac{2e}{m - 2}$: hence $a = \frac{p}{2} + \sqrt{\left(\frac{p^2}{4} + \frac{2e}{m - 2}\right)}$

Ex.

Ex. 2nd. Put $c = 1$, $d = 0$; then $b = a - 1$, and $2bc = 2a - 2 = 2v$: hence $a^2 - pa = 2a - 2 + \frac{2e}{m-2}$, and $a^2 - (2 + p)\, a = \frac{2e}{m-2} - 2$; therefore $a = \frac{2+p}{2} + \sqrt{\left(\frac{(2+p)^2}{4} + \frac{2e}{m-2} - 2\right)}$.

Ex. 3rd. Put $\frac{a}{m} = b = c = d$ &c. Then $2v = a^2 - \frac{a^2}{m}$, and $a^2 - pa = a^2 - \frac{a^2}{m} + \frac{2e}{m-2}$: hence $a^2 - pma = \frac{2me}{m-2}$; therefore $a = \frac{pm}{2} + \sqrt{\left(\frac{p^2m^2}{4} + \frac{2me}{m-2}\right)}$.

PROP. VIII. *Problem.* To resolve a given number e into polygons of a given denomination m by help of the table in Cor. 3. Prop. I.

1st. Write down all the successive values of a, beginning with e and diminishing them progressively by $m - 2$ until the series terminates with 0, or with $p \angle m - 2$ (by Cor. 3. Prop. IX.).

2nd. Under each value of a place the corresponding value of $s - r$, making the series begin with 0, and increase by unity until it ends with s (by the same.).

3rd. Find the limiting value of a by Prop. VII. viz. find a by Examples 1st. and 2nd. and if that given by Ex. 1st. be rational, the value of a assigned by Ex. 2nd. is the limit.

4th. These preparatory steps being completed, take a value of $s - r$ in the work, corresponding to any value of a not less than the minimum, fixed by the limit found by Prop. VII. and find the greatest numbers in the second column of the table to Prop. I. which will produce the same when added together; then if the indices of these numbers, when added in like manner, give the index of $s - r$ in the work, they are also the indices of the constituent polygons: but it will frequently happen (by Cor. 1. Prop. I. and Cor. 1. Prop. II.), that the sum of these indices will be less than (a) corresponding to the assumed value of $s - r$, in which case the deficiency may be made up in the latter sum by the addition of units, because 1 is the index of 0 in the table.

Example 1st. Resolve 17 into pentagons, according to the directions given above: the work will stand thus,

$m = 5$, $e = 17$, $a = 17.\ \ 14.\ \ 11.\ \ 8.\ \ 5.\ \ 2$

$s - r = 0.\ \ 1.\ \ 2.\ \ 3.\ \ 4.\ \ 5.$

In this case the limit of $a = 4{\cdot}72$ &c. (by Prop. VII. Ex. 2.); therefore 5 is the true limit (by Cor. 4. Prop. 6.) and 17 may be resolved by help of five values of $s - r$; viz. 0, 1, 2, 3, 4. The first, 0, resolves it into 17 pentagons: for $17 \times 0 = 0 = s - r$, and $17 \times 1 = 17 = a$. The second, viz. $1 = s - r$ resolves 17 into 12 pentagons of the first and one of the second order: for $12 \times 0 = 0$; $12 \times 1 = 12$; and $1 \times 1 = 1$; $1 \times 2 = 2$; but $0 + 1 = 1 = s - r$; $12 + 2 = 14 = a$. In the same manner the third value of $s - r = 2$, resolves 17 into 7 pentagons of the first order and 2 of the second; for $7 \times 0 = 0$; $7 \times 1 = 7$; and $2 \times 1 = 2$; $2 \times 2 = 4$: but $0 + 2 = 2 = s - r$; $7 + 4 = 11 = a$. Moreover the fourth value of $s - r = 3$, resolves 17 into 2 pentagons of the first and 3 of the second order; for $2 \times 0 = 0$; $2 \times 1 = 2$; and $3 \times 1 = 3$; $3 \times 2 = 6$; but $0 + 3 = 3 = s - r$; $2 + 6 = 8 = a$. Lastly, the fifth value of $s - r = 4$, resolves 17 into 1 pentagon of the second order and 1 of the third: for $1 + 3 = 4 = s - r$; $2 + 3 = 5 = a$.

Ex. 2nd. To resolve $3m - 1$ into 3 polygons of the denomination m.

$$\begin{array}{lcccc} \text{Here } a = e = & 3m - 1, & 2m + 1, & m + 3, & 5 \\ s - r = & 0, & 1, & 2, & 3. \end{array}$$

The last value of $s - r$ or $3 = 3 + 0 + 0$; the sum of whose indices $= 3 + 1 + 1 = 5 = a$; therefore $3m - 1$ is resolved into 3 polygons of the denomination m, viz. into 2 of the first and 1 of the third order.

PROP. IX. Any whole number (e) is a polygon of a given denomination (m); or it may be resolved into polygons of that denomination, the number of which does not exceed m.

For in the first place, suppose $e = 3m - t$; then e may be resolved into $m - t$ polygons of the first order and 2 of the second (by Cor. Prop. III.); therefore if t be greater than 1, $2tm - t$ is not greater than m: but if $t = 1$, e consists of 3 polygons of the denomination m (by Prop. VIII. Ex. 2nd.); but 3 is not greater than m: moreover if $p = 0$; $e = 3m$, which may be resolved into 3 polygons (by Cor. Prop. III.). Again suppose $e > 3m$; then if any one of the values of a, found in Prop. VII. be an integer, it fulfills the conditions of the proposition: For the first makes e a polygon of the denomination m, the second resolves it into 2, one of which is of the first order; and the third value of a resolves e into $m =$ polygons of the same denomination. Lastly, if all the values of a, found by Prop. VII. be fractional, and $e > 3m$, the difference of a (found as in Ex. 2nd. and 3rd. in Prop. VII.) exceeds $m - 2$; therefore a value of a, of the form $a = p + r \times (m - 2)$, falls between these limits, which will resolve e into m polygons of the same denomination, at most (by Cor. 3. Prop. II.).

ARTICLE

ARTICLE II.

On the Binomial Theorem and the Logarithmic Series.

By Mr. KNIGHT.

To the Editor of the Mathematical Repository.

SIR,

M. La Croix, in his Algebra, says that the demonstration of the Binomial Theorem, by the Theory of Combinations, is the most direct that has been yet discovered, in the case of a whole positive exponent.—I conceive the following to be the most direct and simplest *possible,* in that case.

$(a+c)^m = a^m + ma^{m-1}c$ when $m = 1$.
Multiply by $a + c$ and the product is

$$a^{m+1} + (m+1)\,a^m c + ma^{m-1}c^2.$$

If, now, we put m for $m+1$ there results

$(a+c)^m = a^m + ma^{m-1}c + (m-1)\,a^{m-2}c^2$ when m equals 2.

Or, because $m = 2$

$$(a+c)^m = a^m + ma^{m-1}c + \frac{m(m-1)}{2}\,a^{m-2}c^2.$$

Multiply by $a + c$ and we have

$$a^{m+1} + (m+1)\,a^m c + \frac{(m+1)m}{2}\,a^{m-1}c^2 + \frac{m(m-1)}{2}\,a^{m-2}c^3.$$

Put m for $m+1$ and we have, when m is 3,

$$(a+c)^m = a^m + ma^{m-1}c + \frac{m(m-1)}{2}a^{m-1}c^2 + \frac{(m-1)(m-2)}{2}\,{}^{m-2}c^3.$$

Or, because $m = 3$

$$(a+c)^m = a^m + ma^{m-1}c + \frac{m(m-1)}{2}a^{m-1}c^2 + \frac{m(m-1)(m-2)}{2.3}a^{m-2}c^3.$$

Now any person who shall perform the above operation will easily perceive that the same law must continue in the succeeding terms; for, after every multiplication, instead of the coefficient $\frac{m(m-1)(m-2)\ldots\ldots\ldots(m-n-1)}{1.2.3\ldots\ldots\ldots(n+2)}$ we shall

have

have $\frac{m(m-1)\ldots\ldots(m-n)}{1.2.3\ldots\ldots(n+1)}+\frac{m(m-1)\ldots\ldots(m-n-1)}{1.2.3\ldots\ldots(n+2)}$

$$=\frac{m(m-1)\ldots\ldots(m-n)(n+2)+m(m-1)\ldots\ldots(m-n-1)}{1.2.3\ldots\ldots\ldots\ldots(n+2)}$$

$=\frac{(m+1)m(m-1)\ldots\ldots\ldots(m-n)}{1.2.3\ldots\ldots(n+2)}$ which becomes after m is put for $m+1$, $\frac{m(m-1)(m-2)\ldots\ldots\ldots(m-n-1)}{1.2.3\ldots\ldots\ldots(n+2)}$; and has precisely the form it had in the preceding power.

The following is a *general* demonstration of the Binomial Theorem. It is easily shewn that, in all cases, the second term of the expansion of $(a+c)^m$ is $ma^{m-1}c$; let then $(a+c)^m=a^m+ma^{m-1}c+A^{(2)}c^2+A^{(3)}c^3+$, this squared gives

$$(a+c)^{2m}=a^{2m}+2ma^{2m-1}c+\left.\begin{matrix}m^2a^{2m-2}\\+2a^mA^{(2)}\end{matrix}\right\}c^2+\left.\begin{matrix}2ma^{m-1}A^{(2)}\\+\quad 2a^mA^{(3)}\end{matrix}\right\}c^3+\ldots\ldots(\alpha)$$

But

$$(a+c)^{2m}=\left\{(a+c)^2\right\}^m=(a^2+2ac+c^2)^m=a^m\left(a+2c+\frac{c^2}{a}\right)^m$$

$$=a^m\left\{a^m+ma^{m-1}\left(2c+\frac{c^2}{a}\right)+A^{(2)}\left(2c+\frac{c^2}{a}\right)^2+A^{(3)}\left(2c+\frac{c^2}{a}\right)^3+\right\}$$

By comparing the coefficients of the powers of c, here and in equation (α), we find

$$\left.\begin{matrix}m^2a^{2m-2}+2amA^{(2)}=ma^{2m-2}+4a^mA^{(2)}\\2ma^{m-1}A^{(2)}+2amA^{(3)}=4a^{m-1}A^{(2)}+8a^mA^{(3)}\end{matrix}\right\}\text{whence}\left\{\begin{matrix}A=\frac{m(m-1)}{1.2}a^{m-2}\\A^{(3)}=\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}\end{matrix}\right.$$

&c &c. &c.

In the same manner might the coefficients of the higher powers be found; but the law is already plain as far as the third power, and we may easily convince ourselves (as in the last demonstration) that it must hold in all cases of whole positive exponents; if it has place for any one: and with respect to positive or negative fractional exponents:—if we had found the following coefficients as we did $A^{(2)}$ and $A^{(3)}$, they must have had the same form in all cases, being in all found by the same process. We have then always

$$(a+c)^m=a^m+ma^{m-1}c+\frac{m(m-1)}{1.2}a^{m-2}c^2+\&c.$$

By an artifice like the above we easily get the logarithmic series. Let $\log.(1+x)=M^{(1)}x+M^{(2)}x^2+M^{(3)}x^3+$; from the nature of logarithms,

2 log.

$$2\log.(1+x) = \log.(1+x)^2 = \log.(1+2x+x^2);$$

which equation becomes by substitution

$$2M^{(1)}x + 2M^{(2)}x^2 + 2M^{(3)}x^3 + =$$

$$M^{(1)}(2x+x^2) + M^{(2)}(2x+x^2)^2 + M^{(3)}(2x+x^2)^3 +$$

and, comparing the coefficients of the powers of x, we get

$$\left.\begin{array}{l} 2M^{(2)} = M^{(1)} + 4M^{(2)} \\ 2M^{(3)} = 4M^{(2)} + 8M^{(3)} \\ 2M^{(4)} = M^{(2)} + 12M^{(3)} + 16M^{(4)} \\ \quad\&c. \qquad\qquad \&c. \end{array}\right\} \text{whence} \left\{\begin{array}{l} M^{(2)} = -\dfrac{M^{(1)}}{2} \\ M^{(3)} = \dfrac{M^{(1)}}{3} \\ M^{(4)} = -\dfrac{M^{(1)}}{4} \\ \qquad\quad \&c. \end{array}\right.$$

$$\text{And } \log.(1+x) = M^{(1)}\left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \right\}.$$

Papcastle,
October 30*th*, 1809.

THOMAS KNIGHT.

ARTICLE III.

A Proposition to illustrate the 47th Proposition of the Second Book of the Principia. By A. B.

To the Editor of the Mathematical Repository.

SIR,

The 47th Prop. of the second book of the Principia has sometimes been objected to as not sufficiently clear, what is put below is intended, *in some measure*, to obviate these objections; should it appear to you to be of any use in that respect, please to insert it in your very valuable Repository, and you will oblige

Nov. 3*rd*, 1809. Your's, &c. A. B.

PROP. If the particles of air be moved from their places by a force which varies according to any given law; it is required to investigate the law of the force with which they will *continue* to be agitated, upon the supposition that the elasticity of the atmosphere is directly as its density.

Let

Let AE, AF, AG (fig. 1. pl. A.) represent the times since the particles E, F, G, respectively begun to move, and E*e*, F*f*, G*g*, the velocities of each at the same instant of time; and suppose the nature of the curve A*gfe*B to be such, that $r \times \text{AE}^n = \text{E}e$, r and n, being constant quantities. Put $\text{AE} = t$, $\text{EF} = \text{FG} = t'$, $\text{E}e = v$, and $f =$ force with which E is accelerated. The equation of the curve is $rt^n = v$, and the fluxion of the space passed over by any of the particles, $= v\dot{t}$; but, by mechanics, force ×ed into fluxion of the space = velocity ×ed into fluxion of the velocity, that is $f \times v\dot{t} = v\dot{v}$; therefore $f = \dot{v} \div \dot{t} = rn \times t^{n-1}$; the law of the force with which the particles begin to move.

The area AE*e*, or space passed over by the particle E $= \int v\dot{t} = \int rt^n\dot{t} = \frac{r}{n+1} \times t^{n+1}$, therefore the areas AF*f*, AG*g*, or the spaces passed over by the particles F and G $= \frac{r}{n+1} \times (t - t')^{n+1}$ and $\frac{r}{n+1} \times (t - 2t')^{n+1}$ respectively; and the difference of these areas or spaces passed over by the particles, *viz.* E*e*F*f* and F*f*G*g* $= \frac{r}{n+1} \times \left\{(n+1)\,t^n t' - \tfrac{1}{2}n\,(n+1)\,t^{n-1}t'^2 + \&c.\right\}$ and $\frac{r}{n+1} \times \left\{(n+1)\,t^n t' - \tfrac{3}{2}n\,(n+1)\,t^{n-1}t'^2 + \&c.\right\} = r \times (t^n t' - \tfrac{1}{2}nt^{n-1}t'^2 + \&c.)$ and $r \times (t^n t' - \tfrac{3}{2}nt^{n-1}t'^2 + \&c.)$.

Let $d =$ distance between two contiguous particles when in a quiescent state, then it is evident that

$d - r \times (t^n t' - \tfrac{1}{2}nt^{n-1}t'^2 + \&c.)$ and $d - r \times (t^n t' - \tfrac{3}{2}nt^{n-1}t'^2 + \&c.)$ = the respective distances at the end of the time t from the commencement of E's motion. Hence, by the hypothesis, the forces with which E and G act upon F, and the elasticity of the atmosphere, or the force with which they act upon each other in their quiescent state will vary as

$$\frac{1}{d - r(t^n t' - \tfrac{1}{2}nt^{n-1}t'^2 + \&c.)},\quad \frac{1}{d - r(t^n t' - \tfrac{3}{2}nt^{n-1}t'^2 + \&c.)},\ \text{and}\ \frac{1}{d}.$$

Hence the comparative elastic force, or the difference of the forces of E and G upon F : mean elastic force : :

$$\frac{rnt^{n-1}t'^2}{\left\{d-r\left(t^n t'-\frac{1}{2}nt^{n-1}t'^2+\&c.\right)\right\}\left\{d-r\left(t^n t'-\frac{1}{2}nt^{n-1}t'^2+\&c.\right)\right\}} : \frac{1}{d}$$

$:: \dfrac{rnt^{n-1}t'^2}{(d-rt^n t')(d-rt^n t')} : \dfrac{1}{d}$, for t'^2 is infinitely less than the other parts of the expression.

Let s = space through which sound passes in one second of time, then $s : d :: 1'' : t' = d \div s$, therefore by substitution and reduction, the comparative force : mean force :: $rnt^{n-1} : (s - rt^n)^2 \div d$. Put D = latitude of a pulse, or the space passed over by sound while a particle of air vibrates forwards and backwards, and T = time of this vibration, then $s : \mathrm{D} :: 1'' : \mathrm{T} = \mathrm{D} \div s$; if, then, in the last term of the last proportion, T, or $\mathrm{D} \div s$, be substituted for t, we shall have the comparative force : mean elastic force :: $rnt^{n-1} : \left\{s - r\,(\mathrm{D}^n \div s^n)\right\}^2 \div d :: rnt^{n-1} : s^2 \div d$ very nearly; for it is known by experiment that s is very great compared with D, and T, or $\mathrm{D} \div s$ is greater than t, therefore the above proportion may be taken as accurately true. But in the *same constitution* of the atmosphere, the second and fourth terms of the last proportion are constant, hence, the comparative force or that with which the particle F *is moved*, $\propto rnt^{n-1} \propto t^{n-1}$.

In the very same manner it will appear, that if the equation of the curve $AgfeB$, be $rt^n + at^p + bt^q +$ &c. $= v$, and therefore f, or the force with which the particles begin to be agitated vary as $rnt^{n-1} + apt^{p-1} + bqt^{q-1} +$ &c. then the comparative force upon F, or any of the particles, will continue to vary as $rnt^{n-1} + apt^{p-1} + bqt^{q-1} +$ &c.

Hence it is evident, that whatever be the *law* of the force which agitates the particles of air in the beginning of their motion, they will continue to be acted upon according to the *same law*, the constitution of the air being such as is supposed in the Proposition.

Cor. 1. If $n = 1$, and a, b, &c. $= 0$, then the force with which the particles begin and continue to be moved, will vary as $t^{n-1} \propto t^0 \propto 1$, that is, the force for the whole time will be uniform.

Cor.

Cor. 2. From what has been demonstrated above it is evident, that if the particles of air begin to be moved according to the law of a cycloidal pendulum, they will continue to be agitated according to the same law all the time they are in motion. But it is known by experiment, that every elastic chord or fibre of a sounding body, when the vibrations are small, begin and continue their motions according to this law; therefore the particles of air in immediate contact with them, must begin and consequently continue to move according to the same law.

ARTICLE IV.

Investigation of a curious Indeterminate Problem.

By Mr. CUNLIFFE, *R. M. College.*

Let ACB (fig. 2. pl. A.) be a triangle circumscribed by a circle, and let the sides AB, AC and BC be bisected by the perpendiculars mD, nE and pF meeting the circle in D, E, F, and draw the chords as in the figure; and let l denote the intersection of AB, DC. Put $AD = DB = a$, $AE = CE = b$, $BF = CF = c$, and let the diameter of the circumscribing circle be denoted by d. Then by a known property $Dm = a^2 \div d$, $En = b^2 \div d$ and $Fp = c^2 \div d$: whence $Am^2 = Bm^2 = a^2 - (a^4 \div d^2) = (a^2 \div d^2)(d^2 - a^2)$ or $Am = Bm = (a \div d)\sqrt{(d^2 - a^2)}$ and in the same way we shall get $An = nC = (b \div d)\sqrt{(d^2 - b^2)}$, and $Cp = pB = (c \div d)\sqrt{(d^2 - c^2)}$. Now since AD = DB, AE = EC and CF = BF, it is manifest that if the three chords AD, AE and CF are applied in continued succession in the circumference of the circle ACB, they will extend a semi-circle.

Let MPQN (fig. 3. pl. A.) be a semi-circle whose diameter MN is equal to the diameter of the circle circumscribing the triangle ACB, chord $MP = AD = a$, chord $NQ = AE = b$, and draw MQ, NP; then $NP = \sqrt{(d^2 - a^2)}$ and $MQ = \sqrt{(d^2 - b^2)}$: and from what has just been remarked, chord $PQ = CF = c$. And by a known property of a trapezium inscribed in a circle $MP \times NQ + MN \times PQ = NP \times MQ$; or $PQ = CF =$

$$c = \frac{NP \times MQ - MP \times NQ}{MN} = \frac{\sqrt{(d^2 - a^2)} \times \sqrt{(a^2 - b^2)} - ab}{d}.$$

From which conclusion and what has before been deduced, it follows, that the chords AD, AE and CF, and sides of the triangle ACB will be rational when $d^2 - a^2$ and $d^2 - b^2$ are rational squares.

It will also be easy to show in the same circumstance, that the right line Cl which bisects the angle ACB and terminates in in the opposite side AB, together with the perpendicular C*q* will be rational; and consequently the lines bisecting the other two angles and terminating in the opposite sides, together with their corresponding perpendiculars will also in that circumstance be rational. For by the property of a trapezium inscribed in a circle AB × DC = DB × AC + AD × BC = DB × (AC + BC), whence DC. = DB × (AC + BC) ÷ AB, which expression is rational, because by the hypothesis the sides of the triangle ACB and chord AD = DB are rational. The triangles D*l*B and DBC are similar, therefore D*l* : DB :: DB : DC; whence $\mathrm{D}l = \mathrm{DB}^2 \div \mathrm{DC}$ which is also rational, and consequently C*l* the right line bisecting the angle ACB, and terminating in the opposite side AB is also rational. And in the same manner may be shewn that the right lines bisecting the other angles and terminating in the opposite sides are rational.

The perpendiculars from the angles upon the opposite sides will also be rational. For the length of a perpendicular from any angle to its opposite side, is equal to the product of the including sides divided by the diameter of the circumscribing circle; and consequently when all the sides of the triangle and the diameter of the circumscribing circle are rational, the perpendiculars will be rational. It has now been proved, that when $d^2 - a^2$ and $d^2 - b^2$ are both rational squares, then the sides, right-lines from the angles bisecting them and terminating in the opposite sides, and perpendiculars from the angles upon the opposite sides in the triangle ACB will all be expressed by rational numbers. Now it is manifest that d must be expounded by such a number, that its square may be divided into two rational squares two different ways; and this will be the case if $d = (r^2 + s^2) \times (m^2 + n^2)$ for then we may take $a = (r^2 + s^2)(m^2 - n^2)$ and $b = (r^2 - s^2)2mn - 2rs(m^2 - n^2)$: whence $\sqrt{(d^2 - a^2)} = (r^2 + s^2)2mn$; and $\sqrt{(d^2 - b^2)} = (r^2 - s^2)(m^2 - n^2) + 4rsmn$;

$$\text{therefore } c = \frac{\sqrt{d^2 - a^2} \times \sqrt{d^2 - b^2} - ab}{d} = 2rs\,(m^2 + n^2);$$

$$\mathrm{A}m = \mathrm{B}m = \frac{a}{d}\sqrt{d^2 - a^2} = (r^2 + s^2)\frac{2mn\,(m^2 - n^2)}{m^2 + n^2} =$$

$$\frac{(r^2 + s^2)^2 \times 2mn \times (m^2 - n^2)}{(r^2 + s^2) \times (m^2 + n^2)};\ \mathrm{A}n = n\mathrm{C} = \frac{b}{d}\sqrt{d^2 - b^2} =$$

$$\frac{\left\{(r^2 - s^2)\,2mn - 2rs\,(m^2 - n^2)\right\} \times \left\{(r^2 - s^2)(m^2 - n^2) + 4rsmn\right\}}{(r^2 + s^2)\,(m^2 + n^2)}$$

$$\mathrm{C}p = p\mathrm{B} = \frac{c}{d}\sqrt{d^2 - c^2} = \frac{2rs\,(r^2 - s^2)\,(m^2 + n^2)^2}{(r^2 + s^2)\,(m^2 + n^2)} \text{ and}$$

rejecting

rejecting the common denominator $(r^2 + s^2) \times (m^2 + n^2)$, and dividing the remaining expressions by 2 we shall have $mn \times (m^2 - n^2) \times (r^2 + s^2)^2$,

$$\left\{(r^2 - s^2) \times mn - rs(m^2 - n^2)\right\} \times \left\{(r^2 - s^2) \times (m^2 - n^2) + 4rsmn\right\}$$

and $rs \times (r^2 - s^2) \times (m^2 + n^2)^2$, being general expressions for the sides of a triangle ACB, such, that the right-lines bisecting the angles and terminating in the opposite sides, and perpendiculars from the angles upon the opposite sides may all be expressed by rational numbers; in which expressions m, n, r and s may be taken at pleasure.

Example 1. Take $m = 2$, $n = 1$; $r = 3$, $s = 2$; then we shall have 6×169, 504 and 6×125 for the sides of such a triangle: or dividing each by 6, 169, 84 and 125 will express the sides of a triangle with the before-named properties.

Example 2. Let m and n remain as before, and take $r = 4$ and $s = 1$, and we shall have 289, 231 and, 250 for the sides of another triangle that will answer.

ARTICLE V.

Solutions to a Problem in Dynamics.

The following Dynamical Problem has often engaged the attention of Mathematicians. A particular case of it was proposed by John Bernoulli, in Comment. Petrop. for 1727, Tom. 2; but it was not till 1735 that a true solution of the problem was published in the Acta Erud. Liepsick, by the same author.

Herman, indeed, attempted a solution of the case proposed in Comment. Petrop. but fell into some mistakes which rendered his conclusions errorneous. A complete and general solution was afterwards given by D'Alembert in his Traité Dynamique, published in 1744, and since then it has been resolved by several Mathematicians of our own country, viz. by Mr. Atwood in his Treatise on Rectilineal Motion, by Mr. Wildbore in the Gentleman's Diary for 1788, and (either generally or for particular cases) by several of the correspondents to the Repository (see Art. VI. Vol. I. Part II. Art. I. II. and XXI. Vol. II. Part II.). We now insert another solution that we have received from a correspondent; and, as some of our readers may not have the opportunity of consulting the works of Bernoulli and D'Alembert, we shall give the substance of their solutions in the present Article.

PROBLEM

Problem. Suppose a given weight B (fig. 4. pl. A) to descend by the force of gravity along a given curve CGB, and by means of a string passing over a pulley fixed at C, in the same vertical plane, to draw up another given weight A along another curve FAC: required the velocities of the bodies at the points A and B, supposing the motion to commence when they are at the given points F and G.

Solution, *by a Correspondent.*

Let *b*B be the space descended by the weight B in the indefinitely small interval of time $\dot{t}$, and *a*A the contemporary space ascended by the weight A. Draw the vertical lines FR, *al* and BS meeting the horizontal lines AR, GS and *bo* in R, *l*, S and *o*; and with the centre C and radii CA, C*b* describe the arcs A*m*, *bn*.

Put $y =$ SB the vertical distance descended by B,
$x =$ RF the vertical distance ascended by A,
AC $= z$, B*o* $= \dot{y}$, *al* $= \dot{x}$, *am* $=$ B*n* $= \dot{z}$, *a*A $= \dot{r}$,
B*b* $= \ddot{s}$, and $g = 32\frac{1}{6}$ feet, the measure of the accelerating force of gravity.

Then the velocity of the weight A in the direction of the curve *a*A is $\frac{\dot{r}}{\dot{t}}$, (the time being supposed to flow uniformly,) and the variation of the velocity, or the measure of the accelerating force in that direction $= \frac{\ddot{r}}{\dot{t}^2}$. But the weight A cannot ascend in the curve unless its gravity be counteracted, and an additional force impressed upon it sufficient to produce the actual velocity $\frac{\dot{r}}{\dot{t}}$. Now the force of gravity in the direction *a*A is $= g\frac{\dot{x}}{\dot{r}}$, therefore the accelerating force necessary to cause A to ascend in the curve with the velocity $\frac{\dot{r}}{\dot{t}}$ is $= g\frac{\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}$, and the whole motive force acting on A, in the direction *a*A, $=$ $A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right)$, which is equivalent to a motive force $A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right) \times \frac{\dot{r}}{\dot{z}}$, acting in the direction of the string.

Now

Now the motive force, by which A is urged towards C, is derived from the action of the weight B; therefore B will be re-acted upon in the direction CB by an equal motive force; that is, B will be acted upon in the direction BC by a motive force $= A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right) \times \frac{\dot{r}}{\dot{z}}$; which force may be resolved into two others, namely, $A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right) \times \frac{\dot{r}}{\dot{s}}$ acting in the direction of the curve, and retarding the descent of B; and $A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right) \times \frac{\dot{r}\dot{m}}{\dot{z}\dot{s}}$ (m being $= Am = \sqrt{(\dot{s}^2 - \dot{z}^2)}$) acting in a direction perpendicular to the curve, and diminishing or augmenting the pressure.

Therefore, the whole effect of the weight A on the weight B, is equivalent to a motive force $A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right) \times \frac{\dot{r}}{\dot{s}}$, directly opposing the descent of B in the curve.

Again, the velocity with which B actually descends in the curve is $\frac{\dot{s}}{\dot{t}}$, and the variation of that velocity, or the measure of the accelerating force, $= \frac{\ddot{s}}{\dot{t}^2}$. But the force of gravity in the direction of the curve is $= \frac{g\dot{y}}{\dot{s}}$; therefore $\frac{g\dot{y}}{\dot{s}} - \frac{\ddot{s}}{\dot{t}^2}$ is the accelerating force constantly destroyed by the connexion of B with the weight A, and the whole motive force constantly [illegible] $= B \times \left(\frac{g\dot{y}}{\dot{s}} - \frac{\ddot{s}}{\dot{t}^2}\right)$. But the motive force constantly lost by B, must be equal to the motive force constantly retarding its descent;

$$\text{therefore } A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right) \times \frac{\dot{r}}{\dot{s}} = B \times \left(\frac{g\dot{y}}{\dot{s}} - \frac{\ddot{s}}{\dot{t}^2}\right),$$

$$\text{or} \quad \frac{A\dot{r}\ddot{r}}{\dot{t}^2} + \frac{B\dot{s}\ddot{s}}{\dot{t}^2} = g \times (B\dot{y} - A\dot{x}).$$

And, multiplying by 2 and taking the fluents,

$$\text{we have } \frac{A\dot{r}^2}{\dot{t}^2} + \frac{B\dot{s}^2}{\dot{t}^2} = 2g \times (By - Ax);$$

or $Av^2 + Bu^2 = 2g \times (By - Ax)$, no correction being necessary, because the motion is supposed to begin when $x = 0$ and $y = 0$.

Now $v : u :: \dot{r} : \dot{s}$, therefore $u^2 = \frac{v^2 \dot{s}^2}{\dot{r}^2}$, and

$\mathrm{A}v^2 + \frac{\mathrm{B}v^2 \dot{s}^2}{\dot{r}^2} = 2g \times (\mathrm{B}y - \mathrm{A}x)$; hence we have

$v = \dot{r} \sqrt{\frac{2g \times (\mathrm{B}y - \mathrm{A}x)}{\mathrm{A}\dot{r}^2 + \mathrm{B}\dot{s}^2}}$, and $u = \dot{s} \sqrt{\frac{2g \times (\mathrm{B}y - x)}{\mathrm{A}\dot{r}^2 + \mathrm{B}\dot{s}^2}}$, for the velocities required.

COR. 1. Let f and ${}'f$ be the accelerative forces by which the bodies A and B are respectively animated; that is

let $f = \frac{\ddot{r}}{\dot{t}^2}$ and ${}'f = \frac{\ddot{s}}{\dot{t}^2}$ then, $\mathrm{A}\dot{r}f + \mathrm{B}\dot{s}\,{}'f = g \times (\mathrm{B}\dot{y} - \mathrm{A}\dot{x})$.

But $f : {}'f :: \ddot{r} : \ddot{s}$, or ${}'f = \frac{f\ddot{s}}{\ddot{r}}$; therefore $\mathrm{A}\dot{r}f + \frac{\mathrm{B}f\dot{s}\ddot{s}}{\ddot{r}}$ $= g \times (\mathrm{B}\dot{y} - \mathrm{A}\dot{x})$; hence $f = g \times \frac{(\mathrm{B}\dot{y} - \mathrm{A}\dot{x})\ddot{r}}{\mathrm{A}\dot{r}\ddot{r} + \mathrm{B}\dot{s}\ddot{s}}$ and ${}'f$ $= g \times \frac{(\mathrm{B}\dot{y} - \mathrm{A}\dot{x})\ddot{s}}{\mathrm{A}\dot{r}\ddot{r} + \mathrm{B}\dot{s}\ddot{s}}$, as determined by Mr. Atwood.

COR. 2. The motive force $\mathrm{B} \times \left(\frac{g\dot{y}}{\dot{s}} - \frac{\ddot{s}}{\dot{t}^2}\right)$, or $\mathrm{B} \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)$, acting in the direction of the curve, may be resolved into two others, namely, $\mathrm{B} \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)\frac{\dot{s}}{\dot{z}}$, acting in the direction of the string CB, and which is equal to the motive force $\mathrm{A} \times \left(\frac{g\dot{x}}{\dot{r}} + f\right)\frac{\dot{r}}{\dot{z}}$; and $\mathrm{B} \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)\frac{\dot{m}}{\dot{z}}$, acting perpendicularly to the curve, and which is equal to the motive force $\mathrm{A} \times \left(\frac{g\dot{x}}{\dot{r}} + f\right)\frac{\dot{m}\dot{r}}{\dot{z}\dot{s}}$. Now the string is stretched in the direction CA by the motive force $\mathrm{A} \times \left(\frac{g\dot{x}}{\dot{r}} + f\right)\frac{\dot{r}}{\dot{z}}$, and in the direction CB by the equal motive force $\mathrm{B} \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)\frac{\dot{s}}{\dot{z}}$; therefore the whole pressure on the axis of the pulley is equal to the resultant of these two equal forces; that is, the whole pressure on the pulley is =

$$A \times \left(\frac{g\dot{x}}{\dot{r}} + f\right)\frac{\dot{r}}{\dot{z}} \times \frac{\sin ACB}{\sin\frac{1}{2} ACB} = B \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)\frac{\dot{s}}{\dot{z}} \times \frac{\sin ACB}{\sin\frac{1}{2} ACB}.$$

Cor. 3. The pressure of the weight B, against the curve, if it had no connexion with the weight A, would evidently be $= Bg\frac{\dot{\alpha}}{\dot{s}}$, ($\alpha$ being $= SB$): but in consequence of that connexion, the pressure is diminished or augmented by the force $A \times \left(\frac{g\dot{x}}{\dot{r}} + f\right)\frac{\dot{m}\dot{r}}{\dot{z}\dot{s}}$, or $B \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)\frac{\dot{m}}{\dot{z}}$; therefore the whole pressure on the curve, arising from gravity, is $= Bg\frac{\dot{\alpha}}{\dot{s}} \mp B \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)\frac{\dot{m}}{\dot{z}}$; also the pressure arising from the centrifugal force of the body is $= B\frac{u^2}{R}$, (R being the radius of curvature at B); therefore the whole pressure against the curve at B is

$$= B \times \left(\frac{g\dot{\alpha}}{\dot{s}} + \frac{u^2}{R}\right) \mp B \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)\frac{\dot{m}}{\dot{z}}$$

$$= B \times \left(\frac{g\dot{\alpha}}{\dot{s}} + \frac{u^2}{R}\right) \mp A \times \left(\frac{g\dot{x}}{\dot{r}} + f\right)\frac{\dot{m}\dot{r}}{\dot{z}\dot{s}}.$$

And in the same manner, putting $\beta = FR$ and $\dot{n} = \dot{b} = \sqrt{(\dot{r}^2 - \dot{z}^2)}$, the pressure against the curve at A is found

$$= A \times \left(\frac{g\dot{\beta}}{\dot{r}} + \frac{v^2}{R'}\right) \mp A \times \left(\frac{g\dot{x}}{\dot{r}} + f\right)\frac{\dot{n}}{\dot{z}}$$

$$= A \times \left(\frac{g\dot{\beta}}{\dot{r}} + \frac{v^2}{R'}\right) \mp B \times \left(\frac{g\dot{y}}{\dot{s}} - {}'f\right)\frac{\dot{n}\dot{s}}{\dot{z}\dot{r}}$$

Or, because ${}'f = \frac{\dot{u}}{\dot{t}} = \frac{u\dot{u}}{\dot{s}}$, and $f = \frac{\dot{v}}{\dot{t}} = \frac{v\dot{v}}{\dot{r}}$; the pressure at B is $= B \times \left(\frac{g\dot{\alpha}}{\dot{s}} + \frac{u^2}{R}\right) \mp B \times \left(\frac{g\dot{y}}{\dot{s}} - \frac{u\dot{u}}{\dot{s}}\right)\frac{\dot{m}}{\dot{z}}$, and the pressure at $A = A \times \left(\frac{g\dot{\beta}}{\dot{r}} + \frac{v^2}{R'}\right) \mp A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{v\dot{v}}{\dot{r}}\right)\frac{\dot{n}}{\dot{z}}$.

Hence the points where the bodies will quit the curves may be determined by making $\frac{g\dot{z}}{\dot{s}} + \frac{u^2}{R} \mp \left(\frac{g\dot{y}}{\dot{s}} - \frac{u\dot{u}}{\dot{s}}\right)\frac{\dot{m}}{\dot{z}} = 0$,

and $\frac{g\dot{\beta}}{\dot{s}} + \frac{v^2}{R'} \mp \left(\frac{g\dot{x}}{\dot{r}} + \frac{v\dot{v}}{\dot{r}}\right)\frac{\dot{n}}{\dot{z}} = 0$.

The problem may be resolved in a similar manner, when there are more weights than two. For instance, let there be three weights A, B, C, (fig. 5, pl, A,) and let B descend on the curve GB, and draw up the other weights A, C on the curves AF, CH, by means of a string passing over the pullies P, P′.

The same construction and substitution being made as in the preceding case, let Cc, $= \dot{p}$, be the little space described by C in the time $\dot{t}$, and h = the vertical height ascended by C. Then it has been shewn that, the effect of the weight A, in opposing the descent of B, is equal to the motive force $A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right) \times \frac{\dot{r}}{\dot{s}}$; and, it may be shewn in the same way that, the effect of the weight C, in opposing the descent of B, is equal to the motive force $C \times \left(\frac{g\dot{h}}{\dot{p}} + \frac{\ddot{p}}{\dot{t}^2}\right) \times \frac{\dot{p}}{\dot{s}}$; therefore the whole effect of the weights A and C, on the weight B, is equal to the motive force $A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right)\frac{\dot{r}}{\dot{s}} + C \times \left(\frac{g\dot{h}}{\dot{p}} + \frac{\ddot{p}}{\dot{t}^2}\right)\frac{\dot{p}}{\dot{s}}$, directly opposing the descent of B in the curve. Hence we have

$$A \times \left(\frac{g\dot{x}}{\dot{r}} + \frac{\ddot{r}}{\dot{t}^2}\right)\frac{\dot{r}}{\dot{s}} + C \times \left(\frac{g\dot{h}}{\dot{p}} + \frac{\ddot{p}}{\dot{t}^2}\right)\frac{\dot{p}}{\dot{s}} = B \times \left(\frac{g\dot{y}}{\dot{s}} - \frac{\ddot{s}}{\dot{t}^2}\right),$$

$$\text{or } A\frac{\dot{r}\ddot{r}}{\dot{t}^2} + B\frac{\dot{s}\ddot{s}}{\dot{t}^2} + C\frac{\dot{p}\ddot{p}}{\dot{t}^2} = g \times (B\dot{y} - A\dot{x} - C\dot{h});$$

and, taking the fluents, we get

$$A\frac{\dot{r}^2}{\dot{t}^2} + B\frac{\dot{s}^2}{\dot{t}^2} + C\frac{\dot{p}^2}{\dot{t}^2} = 2g \times (By - Ax - Ch),$$

or (putting ω' = the velocity of C)

$$Av^2 + Bu^2 + C\omega'^2 = 2g \times (By - Ax - Ch).$$

But $v = \frac{\dot{r}}{\dot{s}}u = \frac{\dot{r}}{\dot{p}}\omega'$, therefore

$$Av^2 + Bv^2\frac{\dot{s}^2}{\dot{r}^2} + Cv^2\frac{\dot{p}^2}{\dot{r}^2} = 2g \times (By - Ax - Ch),$$

$$\text{and } v = \dot r \sqrt{\frac{2g \times (By - Ax - Ch)}{A\dot r^2 + B\dot s^2 + C\dot p^2}}$$

$$u = \dot s \sqrt{\frac{2g \times (By - Ax - Ch)}{A\dot r^2 + B\dot s^2 + C\dot p^2}}$$

$$w' = \dot p \sqrt{\frac{2g \times (By - Ax - Ch)}{A\dot r^2 + B\dot s^2 + C\dot p^2}}.$$

Again, let there be four weights A, B, C, D (fig. 6, pl. A,) and let B and D descend on the curves GB, KD, and draw up the weights A and C on the curves FA, HC. Let $Dd = \dot q$, be the little space descended by D in the time $\dot t$, $\varrho = bb' = cc'$, $\rho = cc'' = D\delta$, and $k =$ the vertical height descended by D. Then the effect of the weight A, on the weight B, is equal to the motive force $A \times \left(\frac{g\dot x}{\dot r} + \frac{\ddot r}{\dot t^2}\right)\frac{\dot r}{\dot s}$. But the motive force constantly lost by B, in consequence of its connexion with A and C, is $= B \times \left(\frac{g\dot y}{\dot s} - \frac{\ddot s}{\dot t^2}\right)$; therefore the motive force lost by B, in consequence of its connexion with C, is $= B \times \left(\frac{g\dot y}{\dot s} - \frac{\ddot s}{\dot t^2}\right) - A \times \left(\frac{g\dot x}{\dot r} + \frac{\ddot r}{\dot t^2}\right)\frac{\dot r}{\dot s}$, which is equivalent to a motive force $B \times \left(\frac{g\dot y}{\dot s} - \frac{\ddot s}{\dot t^2}\right)\frac{\dot s}{\dot\varrho} - A \times \left(\frac{g\dot x}{\dot r} - \frac{\ddot r}{\dot t^2}\right)\frac{\dot r}{\dot\varrho}$ acting in the direction of the string BP′. Let this force be denoted by F; then the weight C, is urged towards P′, by a motive force F, which is equivalent to a motive force $\frac{\dot\varrho}{\dot p} \times F$, acting in the direction of the curve Cc. Again, the motive force gained by C, in consequence of its connexion with B and D, is $= C \times \left(\frac{g\dot h}{\dot p} + \frac{\ddot p}{\dot t^2}\right)$; therefore $C \times \left(\frac{g\dot h}{\dot p} + \frac{\ddot p}{\dot t^2}\right) - \frac{\dot\varrho}{\dot p} \times F$, is the motive force gained by C, in consequence of its connexion with D, and which is equivalent to a motive force $\frac{\dot p}{\dot\rho} \times C \times \left(\frac{g\dot h}{\dot p} + \frac{\ddot p}{\dot t^2}\right) - \frac{\dot\varrho}{\dot\rho} \times F$, acting in the direction of the string CP″. Therefore the weight D, is urged towards P″, by a motive force $\frac{\dot p}{\dot\rho} \times C \times \left(\frac{g\dot h}{\dot p} + \frac{\ddot p}{\dot t^2}\right) - \frac{\dot\varrho}{\dot\varrho} \times F$, which is equivalent to a motive force $\frac{\dot p}{\dot q}\times C\times\left(\frac{g\dot h}{\dot p}+\frac{\ddot p}{\dot t^2}\right)-\frac{\dot s}{\dot q}\times F$, acting in the direction of the curve D*d*, and directly opposing the descent of D. Therefore, the whole effect of the weights A, B, C in retarding the descent of D, is equivalent to the motive force $\frac{\dot p}{\dot q}\times C\times\left(\frac{g\dot h}{\dot p}+\frac{\ddot p}{\dot t^2}\right)-\frac{\dot s}{\dot q}\times F$, and which must be equal to $D\times\left(\frac{g\dot k}{\dot q}-\frac{\ddot q}{\dot t^2}\right)$, the motive force constantly lost by D.

Therefore, putting for F its proper value, we have

$$\frac{\dot p}{\dot q}\times C\times\left(\frac{g\dot h}{\dot p}+\frac{\ddot p}{\dot t^2}\right)-\frac{\dot s}{\dot q}\times B\times\left(\frac{g\dot y}{\dot s}-\frac{\ddot s}{\dot t^2}\right)+\frac{\dot r}{\dot q}\times A\times\left(\frac{g\dot x}{\dot r}+\frac{\ddot r}{\dot t^2}\right)=D\times\left(\frac{g\dot k}{\dot q}-\frac{\ddot q}{\dot t^2}\right)$$

or $$A\frac{\dot r\ddot r}{\dot t^2}+B\frac{\dot s\ddot s}{\dot t^2}+C\frac{\dot p\ddot p}{\dot t^2}+D\frac{\dot q\ddot q}{\dot t^2}=g\times(B\dot y+D\dot k-A\dot x-C\dot h)$$

and, taking the fluents,

$$A\frac{\dot r^2}{\dot t^2}+B\frac{\dot s^2}{\dot t^2}+C\frac{\dot p^2}{\dot t^2}+D\frac{\dot q^2}{\dot t^2}=2g\times(By+Dk-Ax-Ch)$$

or $Av^2+Bu^2+C\omega^2+D\omega'^2=2g\times(By+Dk-Ax-Ch)$.

By proceeding in this manner the problem may be resolved for any number of weights whatever, and the general result will be, that "The sum of the products of each body, by the square of its velocity, is equal $2g$ multiplied into the products of each body by the space it has perpendicularly descended." The spaces being taken negatively when the bodies ascend, and positively when they descend.

This is all that can be determined relating to the velocities of the bodies, unless the nature of the curves be known so as to enable us to find the relation of $\dot r$, $\dot s$, $\dot p$, &c. We shall take a few particular examples, for the purpose of applying the preceding formula.

Ex. 1. To find the velocities of the bodies, when B descends on the inclined plane DGB (fig. 7, pl. A), and draws up the weight A, on the inclined plane DAF, by means of a string passing over a pulley at C.

Suppose the motion to commence when the bodies are at F and G, and from C, let CH and CL be drawn perpendicular to the given planes.

Put CH $= a$, CL $= b$, FH $= c$, GL $= d$, the length of the string $= l$; and let, as in the general problem, FA $= r$, GB $= s$, AC $= z$, FR $= x$, and BS $= y$. Then AH $= \sqrt{(z^2 - a^2)}$, $r = c - \sqrt{(z^2 - a^2)}$, $\dot{r} = \frac{-z\dot{z}}{\sqrt{(z^2 - a^2)}}$, BL $= \sqrt{((l - z)^2 - b^2)}$, $s = \sqrt{((l - z)^2 - b^2)} - d$, and $\dot{s} = \frac{-(l - z)\dot{z}}{\sqrt{((l - z)^2 - b^2)}}$. These values of $\dot{r}$ and $\dot{s}$, being substituted in the general formula for the values of v and u, we get, after proper reduction,

$$v = z\sqrt{2g\,((l-z)^2 - b^2)} \times \sqrt{\frac{\mathrm{B}y - \mathrm{A}x}{\mathrm{A}z^2((l-z)^2 - b^2) + \mathrm{B}(l-z)^2(z^2 - a^2)}}$$

and $u = (l-z)\sqrt{2g(z^2 - a^2)} \times \sqrt{\frac{\mathrm{B}y - \mathrm{A}x}{\mathrm{A}z^2((l-z)^2 - b^2) + \mathrm{B}(l-z)^2(z^2 - a^2)}}$.

But to exterminate x and y, let $\alpha =$ the angle FAR, and $\beta =$ the angle BGS; then $x = r \sin\alpha = (c - \sqrt{(z^2 - a^2)}) \sin\alpha$, and $y = s\sin\beta = (\sqrt{((l-z)^2 - b^2)} - d)\sin\beta$; and, by substitution, we have

$$v = z\sqrt{2g((l-z)^2 - b^2)} \times \sqrt{\frac{\mathrm{B}(\sqrt{((l-z)^2 - b^2)} - d)\sin\beta - \mathrm{A}(c - \sqrt{(z^2 - a^2)}\sin\alpha}{\mathrm{A}z^2((l-z)^2 - b^2) + \mathrm{B}(l-z)^2(z^2 - a^2)}},$$

and $u = (l-z)\sqrt{2g(z^2 - a^2)} \times \sqrt{\frac{\mathrm{B}(\sqrt{((l-z)^2 - b^2)} - d)\sin\beta - \mathrm{A}(c - \sqrt{(z^2 - a^2)})\sin\alpha}{\mathrm{A}z^2((l-z)^2 - b^2) + \mathrm{B}(l-z)^2(z^2 - a^2)}}$,

for the velocities required in terms of z and known quantities.

When the pulley is at D, the intersection of the inclined planes, then $a = 0$, and $b = 0$; and we have, $v = u =$

$$\sqrt{2g} \times \sqrt{\frac{\mathrm{B}(l - z - d)\sin\alpha - \mathrm{A}(c - z)\sin\beta}{\mathrm{A} + \mathrm{B}}} =$$

$\sqrt{\frac{2g}{\mathrm{A} + \mathrm{B}}} \times \sqrt{(\mathrm{B}r \sin\alpha - \mathrm{A}s \sin\beta)}$. And when the plane BD is vertical, $v = u = \sqrt{\frac{2g}{\mathrm{A} + \mathrm{B}}} \times \sqrt{(\mathrm{B}r \sin\alpha - \mathrm{A}s)}$.

Again, when the plane FD is horizontal, and the weight B descends in a vertical direction; then $x = 0$, $\alpha = 0$, $b = 0$, and $\sin\beta = 1$; and we have

$v = z \times \sqrt{\frac{2g \times \mathrm{B}(l - z - d)}{\mathrm{A}z^2 + \mathrm{B}(z^2 - a^2)}}$, and $u = \sqrt{(z^2 - a^2)} \times \sqrt{\frac{2g \times \mathrm{B}(l - z - d)}{\mathrm{A}z^2 + \mathrm{B}(z^2 - a^2)}}$:

The method of calculation will be nearly the same when there are more bodies than two moving on inclined planes, but the expressions for the velocities are too complex to be put down here.

EX. 2. To find the velocities of the bodies when B descends in a vertical direction, and draws up the body A in a circle FAD; or, which is the same thing, when the body A is fastened to an inflexible rod of a given length, without weight, moveable about the centre O, and drawn up by the weight B, by means of a string passing over a pulley at C. Fig. 8, pl. A.

Let O be the centre of the circle; and draw AH, perpendicular to CO, meeting the horizontal line OD in K. Also draw AI and FN perpendicular to OD. Put OC $= a$, radius of the circle $= c$, OH $= w$, FA $= r$, GB $= s$, and AC $= z$; then, by the property of the circle, $\dot{r} = \frac{c\dot{w}}{\sqrt{(c^2 - w^2)}}$; also $z^2 = \text{CH}^2 - \text{OH}^2 + \text{OA}^2 = a^2 - 2aw + c^2$, and $\dot{z} = -\frac{a\dot{w}}{z} = -\dot{s}$. These values of $\dot{r}$ and $\dot{s}$, being substituted in the general formula for v and u, we have, after proper reduction,

$$v = cz \times \sqrt{\frac{2g \times (\text{B}y - \text{A}x)}{\text{A}c^2z^2 + \text{B}a^2(c^2 - w^2)}}$$

$$\text{and } u = \frac{a}{\sqrt{(c^2 - w^2)}} \times \sqrt{\frac{2g \times (\text{B}y - \text{A}x)}{\text{A}c^2z^2 + \text{B}a^2(c^2 - w^2)}}.$$

But to exterminate x and y; let $\alpha =$ the angle COD, $d =$ FC, $e =$ FN; then HK $= w \tan\alpha$, HA $= \sqrt{(c^2 - w^2)}$, KA $= \sqrt{(c^2 - w^2)} - w\tan\alpha$, AI $= \cos\alpha(\sqrt{(c^2 - w^2)} - w\tan\alpha)$ and $x = e - \cos\alpha(\sqrt{(c^2 - w^2)} - w\tan\alpha)$. Also because CF + GC $=$ AC + CG + GB, $d = z + y$, or $y = d - z = d - \sqrt{(a^2 - 2aw + c^2)}$; and we have

$$v = cz\sqrt{2g} \times \sqrt{\frac{\text{B}(d - z) - \text{A}(e - \cos\alpha(\sqrt{(c^2 - w^2)} - w\tan\alpha))}{\text{A}c^2z^2 + \text{B}a^2(c^2 - w^2)}}$$

$$\text{and } u = a\sqrt{\frac{2g}{c^2 - w^2}} \times \sqrt{\frac{\text{B}(d - z) - \text{A}(e - \cos\alpha(\sqrt{(c^2 - w^2)} - w\tan\alpha))}{\text{A}c^2z^2 + \text{B}a^2(c^2 - w^2)}}$$

where $z = \sqrt{(a^2 - 2aw + c^2)}$, or $w = \frac{a^2 + c^2 - z^2}{2a}$.

When the pulley is at D, and the arch FD is a quadrant, as in the prize question in the Ladies Diary for 1810, then $\alpha = 0$, $w =$ OI, $a = c = e$, $d = c\sqrt{2}$, and we have

$$v = z\sqrt{2g} \times \sqrt{\frac{\text{B}(d - z) - \text{A}(c - \sqrt{(c^2 - w^2)})}{\text{A}z^2 + \text{B}(c^2 - w^2)}}$$

$$\text{and } u = a\sqrt{\frac{2g}{c^2 - w^2}} \times \sqrt{\frac{\text{B}(d - z) - \text{A}(c - \sqrt{(c^2 - w^2)})}{\text{A}z^2 + \text{B}(c^2 - w^2)}}.$$

Or, substituting for z and w their proper values,

$$v = \sqrt{4gc} + \sqrt{\frac{\text{B}(c\sqrt{2} - \sqrt{(2c^2 - 2cw)}) - \text{A}(c - \sqrt{(c^2 - w^2)})}{2c\text{A} + \text{B}(c + w)}}$$

$$= 2c\sqrt{2g} \times \sqrt{\frac{\text{B}(c\sqrt{2} - z) - \text{A}(c - \frac{z}{2c}\sqrt{(4c^2 - z^2)})}{\text{A} \times 4c^2 + \text{B}(4c^2 - z^2)}},$$

which agrees with the Diary solution in the particular instant, only, when $z = 0$, or the body arrives at D.

When the weight A, descends from D, in the circle, and draws up the weight B, the problem is the same as quest. 545.

Gents. Diary for 1785. In this case $x = \text{AI} = \sqrt{(c^2-w^2)}$, y (the space ascended by B) $= \text{AD} = z$, $z = \sqrt{(2c^2-2cw)}$, and

$$v = z\sqrt{\frac{2g\times(\text{A}x-\text{B}y)}{\text{A}z^2+\text{B}(c^2-w^2)}} = \sqrt{2c}\times\sqrt{\frac{2g\times(\text{A}\sqrt{(c^2-w^2)}-\text{B}\sqrt{(2c^2-2cw)})}{2\text{A}c+\text{B}(c+w)}}.$$

agreeing with the corrected expression for the velocity, given in the Diary for 1787.

Ex. 3. To find the velocities of the bodies, when B descends in a given circle D′GB, and draws up the weight A, which moves in another given circle FAD; or, which is the same thing, when the bodies are fastened to the ends of two inflexible slender rods, without weight, moveable about the given points O, O′, and connected by a string passing over a pulley at C. Fig. 9, pl. A.

Let O, O′ be the centres of the given circles, and draw AH, and BH′ perpendicular to CO and CO′, meeting the horizontal lines OD, O′D′ in K and K′; and suppose the motion to commence when the bodies are at F and G.

Put $a = \text{OC}$, $b = \text{O'C}$, $c =$ the radius OD, $c' =$ the radius O′D′, $z = \text{AC}$, $r = \text{FA}$, $s = \text{GB}$, $\text{OH} = w$, $\text{O'H'} = \tilde{w}$, $x = \text{FR}$ and $y = \text{SB}$.

Then $\dot{r} = \dfrac{c\dot{w}}{\sqrt{(c^2-w^2)}}$, $z^2 = a^2 - 2aw + c^2$, $\dot{z} = -\dfrac{a\dot{w}}{z}$, $\dot{s} = \dfrac{c'\dot{\tilde{w}}}{\sqrt{(c'^2-\tilde{w}^2)}}$

$\text{CB}^2 = (l-z)^2 = b^2 - 2b\tilde{w} + c'^2$, $(l-z)^{\cdot} = -\dot{z} = \dfrac{b\dot{\tilde{w}}}{l-z}$; hence $\dfrac{a\dot{w}}{z} = \dfrac{b\dot{\tilde{w}}}{l-z}$, or $\dot{\tilde{w}} = \dfrac{a(l-z)\dot{w}}{bz}$, and therefore $\dot{s} = \dfrac{c'a(l-z)\dot{w}}{bz\sqrt{(c'^2-\tilde{w}^2)}}$.

These values of $\dot{r}$ and $\dot{s}$, being substituted in the general formula for v and u, we shall have

$$v = cbz\sqrt{\frac{2g\times(\text{B}y-\text{A}x)\times(c'^2-\tilde{w}^2)}{\text{A}c^2b^2z^2(c'^2-\tilde{w}^2)+\text{B}c'^2a^2(l-z)^2(c^2-w^2)}}$$

$$u = c'a(l-z)\sqrt{\frac{2g\times(\text{B}y-\text{A}x)\times(c^2-w^2)}{\text{A}c^2b^2z^2(c'^2-\tilde{w}^2)+\text{B}c'^2a^2(l-z)^2(c^2-w^2)}}.$$

To exterminate x and y, let $\alpha =$ the angle COD, $\beta =$ the angle CO′D′, $e = \text{FN}$, $e' = \text{GN'}$; then, as in the last example, $x = e - \cos\alpha\,(\sqrt{(c^2-w^2)} - w\tan\alpha)$, and $y = \cos\beta\,(\sqrt{c'^2-\tilde{w}^2}) - \tilde{w}\tan\beta) - e'$, and these values being substituted in the above expressions, we shall have the velocities required in terms of z, w and $\tilde{w}$; any two of which may be exterminated by means of the equations $z^2 = a^2 - 2aw + c^2$, and $(l-z)^2 = b^2 - 2b\tilde{w} + c'^2$.

Ex. 4. A body A, being fastened to an inflexible rod of a given length, without weight, moveable about the point O, is drawn up by means of the weights B, C fastened to the ends of the strings APB, AP′C, which pass over the fixed pullies P, P′ (fig. 10, pl. A). It is required to find the velocities of the bodies, Draw AH and AH′ perpendicular to OP and OP′, meeting the horizontal line OD in K and K′; and suppose the bodies to be at F, G and S at the commencement of the motion. Put $OP = a$, $OP' = b$, radius $OA = c$, $OH = w$, $OH' = \tilde{\omega}$, $AP = z$, $AP' = z'$, $AF = r$, the height RF, ascended by A, $= x$, the space GB, descended by B, $= y\ (= s)$, and the space descended by $C = h\ (= p)$.

Then $\dot{r} = -\frac{c\dot{w}}{\sqrt{(c^2-w^2)}}$, $z^2 = a^2 - 2aw + c^2$, $\dot{z} = -\frac{a\dot{w}}{z} = -\dot{s}$,

$z'^2 = b^2 - 2b\tilde{\omega} + c^2$, $\dot{z}' = -\frac{b\dot{\tilde{\omega}}}{z'} = -\dot{p}$.

But $-\frac{c\dot{w}}{\sqrt{(c^2-w^2)}} = \dot{r} = \frac{c\dot{\tilde{\omega}}}{\sqrt{(c^2-\tilde{\omega}^2)}}$;

therefore $\dot{\tilde{\omega}} = -\frac{\dot{w}\sqrt{(c^2-\tilde{\omega}^2)}}{\sqrt{(c^2-w^2)}}$, and $\frac{b\dot{\tilde{\omega}}}{z'} = \frac{b\dot{w}}{z'} \times \sqrt{\frac{c^2-\tilde{\omega}^2}{c^2-w^2}} = \dot{p}$;

hence $\frac{\dot{s}}{\dot{r}} = \frac{a\sqrt{(c^2-w^2)}}{cz}$, and $\frac{\dot{p}}{\dot{r}} = \frac{b\sqrt{(c^2-\tilde{\omega}^2)}}{cz'}$; and substituting these values of $\frac{\dot{s}}{\dot{r}}$ and $\frac{\dot{p}}{\dot{r}}$ in the general formula for the velocities, we have

$$Av^2 + Bv^2 \times \frac{a^2(c^2-w^2)}{c^2z^2} + Cv^2 \times \frac{b(c^2-\tilde{\omega}^2)}{c^2z'^2} = 2g \times (By - Ax + Ch),$$

from whence we get

$$v = czz'\sqrt{\frac{2g \times (By - Ax + Ch)}{Ac^2z^2z'^2 + Ba^2(c^2-w^2)z'^2 + Cb^2(c^2-\tilde{\omega}^2)z^2}}.$$

$$\text{Then } u = az'\sqrt{(c^2-w^2)} \times \sqrt{\frac{2g \times (By - Ax + Ch)}{Ac^2z^2z'^2 + Ba^2(c^2-w^2)z'^2 + Cb^2(c^2-\tilde{\omega}^2)z^2}}$$

$$\text{and } w' = bz\sqrt{(c^2-\tilde{\omega}^2)} \times \sqrt{\frac{2g \times (By - Ax + Ch)}{Ac^2z^2z'^2 + Ba^2(c^2-w^2)z'^2 + Cb^2(c^2-\tilde{\omega}^2)z^2}}.$$

To exterminate x, y and h; put $\alpha =$ the angle POD, $\beta =$ the angle P′OD, $e =$ FN, $d =$ FP, $d' =$ FP′. Then, as in the last example, we find

$$AI = \cos\alpha(\sqrt{(c^2-w^2)} - w\tan\alpha) = \cos\beta(\sqrt{(c^2-\tilde{\omega}^2)} + \tilde{\omega}\tan\beta)$$

$x = e - \cos\alpha(\sqrt{(c^2-w^2)} - w\tan\alpha)$, $y = d - z$, and $h = d' - z'$; these values of x, y and h being substituted in the above expressions, we shall have the velocities required, in terms of z, z', w and $\tilde{\omega}$; any three of which may be exterminated by means of the equations $z^2 = a^2 - 2aw + c^2$, $z'^2 = b^2 - 2b\tilde{\omega} + c^2$, and the equation expressing the relation between w and $\tilde{\omega}$, viz. $\cos\alpha(\sqrt{(c^2-w^2)} - w\tan\alpha) = \cos\beta(\sqrt{(c^2-\tilde{\omega}^2)} + \tilde{\omega}\tan\beta)$.

The solutions of these problems will not be materially different when the inertia of the pullies and the weights of the rods, &c.

are taken into account, but I shall not prosecute the subject any farther at present.

J. L.

* * *

Two solutions by John Bernoulli; being extracts from a Dissertation published in the Acta Erud. Leipsick, for 1735, and inserted at No. cxlv, Tom. iii, of his works, printed at Lausanne.

1st. An indirect solution deduced from the principle virium vivarum. Fig. 4, pl. A.

Through C let the horizontal line MCE be drawn and let aCb be a position of the string indefinitely near to ACB. From the points B, A draw the vertical lines BE, AD; and with the centre C and radii Cb, CA describe the arcs bn, Am. Also through the points b, A draw bo, Al parallel to the horizon meeting the vertical lines BE, al, in o and l.

Put $CB = x$, $Bn = am = dx$, $Bo = dy$, $bB = ds$, and let z be the height through which a heavy body, at liberty to descend freely, must fall to acquire the same velocity that the weight B has in the curve at B. It is required to find the altitude z.

Let $Aa = dr$ and $la = dq$; then since the velocity at B is proportional to the square root of the altitude z, the velocity may be put $= \sqrt{z}$; and then (because $bB : aA$: : velocity at B : velocity at A) the velocity at A will be $= \frac{dr}{ds}\sqrt{z}$. Suppose the weight B to begin to descend from G, and the weight A to ascend from F, and draw GS parallel to the horizon meeting the vertical line BE in S, and produce Al to meet the vertical line FM in R; then SB is the vertical height descended by B, and FR the vertical height ascended by A, and by the principle conservatio virium vivarum we have

$$SB \times B - FR \times A = z \times B + \frac{dr^2}{ds^2} z \times A;$$

$$\text{hence } z = \frac{(SB \times B - FR \times A)\, ds^2}{B ds^2 + A\, dr^2},$$

or (writing y for SB, and q for FR)

$$z = \frac{(By - Aq)\, ds^2}{B ds^2 + A dr^2}.$$

2. A direct solution from the pure principles of Mechanics.

Let T be the tension of the string which connects the given weights, and $g =$ the force of gravity, or the natural accelerative force by which the bodies are animated in a vertical direction. Then if the masses of the bodies be denoted by B and A, their absolute weights will be denoted by gB and gA, and the forces by which they are urged in the oblique directions *b*B, *a*A will be denoted by $gB\frac{dy}{ds}$ and $gA\frac{dq}{dr}$ respectively. Also the force of tension T when referred to the oblique directions *b*B, *a*A, produces in the direction *b*B the tension $T \times \frac{dx}{ds}$, acting contrary to the force of gravity $gB\frac{dy}{ds}$; and in the direction *a*A the tension $T \times \frac{dx}{dr}$, acting contrary to the force of gravity $gA\frac{dq}{dr}$.

Hence $gB \times \frac{d}{ds} - T \times \frac{dx}{ds}$ is the motive force by which B is urged along *b*B, and $T \times \frac{dx}{dr} - gA\frac{dq}{dr}$ the motive force by which A is urged along *a*A; and the motive forces being divided by the respective masses B and A, we have $g \times \frac{dy}{ds} - \frac{Tdx}{Bds}$ for the force by which B is accelerated, and $\frac{Tdx}{Adr} - g \times \frac{dq}{dr}$ for the force by which A is accelerated.

By dynamics $PdS = VdV$ (P being the accelerative force, dS the element of the space, and V the velocity), hence if u be put for the velocity of B, and v for that of A, we have these two equations

$$\left(g \times \frac{dy}{ds} - \frac{Tdx}{Bds}\right) ds, \text{ or } gdy - \frac{Tdx}{B} = udu.$$

$$\text{and } \left(\frac{Tdx}{Adr} - g \times \frac{dq}{dr}\right) dr, \text{ or } \frac{Tdx}{A} - gdq = vdu.$$

Hence, by integrating, we get $gy - \frac{1}{B} \int Tdx = \frac{1}{2}u^2$, and $\frac{1}{A} \int Tdx - gq = \frac{1}{2}v^2$. From the first $\int Tdx = gBy - \frac{1}{2}Bu^2$, and from the second $\int Tdx = gAq + \frac{1}{2}Av^2$,

hence $gBy - \frac{1}{2}Bu^2 = gAq + \frac{1}{2}Av^2$.

But by hypothesis, z designates the altitude through which a heavy body must descend to acquire the velocity u, therefore by dynamics $gdz = udu$; hence $gz = \frac{1}{2}u^2$, and because $\frac{u^2}{v^2} = \frac{ds^2}{dr^2}$, we have $\frac{1}{2}v^2 = \frac{dr^2}{2ds^2}u^2 = g\frac{dr^2}{ds^2}z$. Substituting these values for $\frac{1}{2}u^2$ and $\frac{1}{2}v^2$ in the equation $gBy - \frac{1}{2}Bu^2 = gAq + \frac{1}{2}Av^2$ we get

$$By - Bz = Aq + \frac{dr^2}{ds^2} Az,$$ which being reduced,

gives $$z = \frac{(By - Aq)ds^2}{Bds^2 + Adr^2}$$ as in the first solution.

COR. 1. From this solution the force of tension T may be easily found. For since $\int Tdx = gBy - \frac{1}{2} Bu^2 = gBy - gBz$, by taking fluxions and dividing by dx, we find $T = gB\frac{(dy - dz)}{dx} = g\,B\,\frac{d(y - z)}{dx} =$ (substituting for z its value)

$$\frac{gB}{dx} \times d\left(y - \frac{Byds^2 - Aqds^2}{Bds^2 + Adr^2}\right) = \frac{gB}{dx} \times d\left(\frac{Aqds^2 + Aydr^2}{Bds^2 + Adr^2}\right).$$

COR. 2. In the particular case proposed in the Comment. Petropolit, when CAF is a vertical line and the body begins to descend from C, the elements dr, dq, and dx are equal, and $q = x$: hence if dx^2 be written for dr^2 and x for q, we have

$$z = \frac{(By - Ax)ds^2}{Bds^2 + Adx^2} \text{ and } T = \frac{gBA}{dx} \times d\left(\frac{xds^2 + ydx^2}{Bds^2 + Adx^2}\right).$$

* * *

A solution by D'Alembert, taken from his Traité de Dynamique.

Fig. 2, pl. A. A body P descending along a curve CB, draws after it another body F by means of a string PCF which passes over a pulley: It is required to find the velocity of each of the bodies.

Let Pp be the element passed over in an instant by the body P, and $Ff = pV$ the element passed over by the other body in the same instant. Then in the succeeding instant, the bodies, if there was nothing to stop them, would pass over $p\pi = Pp$, and $f\varphi = Ff$. But because of the resistance of the string and its inextensibility, the line $p\pi$ is passed over by the body P in an instant different from the first, and in this instant the body F arrives at a point $\bar{\omega}$, such that $\bar{\omega}C + C\pi = fC + Cp$.

Supposing now that, in the time the body P is in passing over $p\pi$, it would naturally pass over pi, and that the body F would naturally describe $f\omega$; also that the gravity of the body F in the same instant would make it descend through the vertical line $\omega\nu$, and that that part of the gravity of the body P which acts in the direction pi, would cause it to descend throught the line il: then if we take $po = \pi l$, it follows from our principle, that the body F animated by the accelerative force represented by $\bar{\omega}\nu$ will make an equilibrium with the body P animated by the force po, and that part of its gravity and the centrifugal force which is perpendicular to the curve Cp, or in the direction pZ. Or if the line $o\alpha$ be drawn from the point o to meet Cp produced in α, and αu be drawn, then the rest of the force in the direction pZ being destroyed by the resistance of the curve, it is clear that we shall have an equilibrium when

$$F . \varpi v = P . p\pi$$

Let u be the velocity of the body P, p its absolute gravity, g that of the body F, $Ff = dx$, $NP = y$, $Pp = ds$, $\pi i = \alpha$; then we have $il = \frac{pdy . ds^2}{ds . u^2}$, $\omega v = g\frac{ds^2}{u^2}$ and $\varphi\omega : Ff :: i\pi : Pp$, that is to say, $\varphi\omega = \frac{\alpha dx}{ds}$; let $\varpi v = n$, then we find $\varphi\varpi$ or $-ddx = \frac{\alpha dx}{ds} + g\frac{ds^2}{u^2} - n$; also because $F . \varpi v = P . p\pi$, we have $F . n = P . \frac{po . ds}{dx} = \frac{Pds}{dx}\left(\frac{pdy . ds^2}{u^2 ds} - \alpha\right)$.

Therefore $Fddx + \frac{Fgds^2}{u^2} + \frac{F\alpha dx}{ds} = -\frac{P\alpha ds}{dx} + \frac{Ppds^2dy}{u^2dx}$;

and consequently $\alpha = \frac{-Fddx - \frac{Fgds^2}{u^2} + \frac{Ppds^2dy}{u^2dx}}{Fdx^2 + Pds^2} \times dsdx$;

but $\frac{\alpha}{ds} = \frac{du}{u}$, therefore $\frac{-Fu^2dxddx - Fgds^2dx + Ppds^2dy}{Fdx^2+Pds^2} = udu$; or $2udu . \frac{(Fdx^2 + Pds^2)}{Pds^2} + \frac{2Fu^2dxddx}{Pds^2} = 2pdy - \frac{2Fgdx}{P}$, the complete integral of which (supposing $u = 0$ when x and $y = 0$) is $\frac{u^2(Pds^2 + Fdx^2)}{Pds^2} = 2py - \frac{2Fgx}{P}$; and making $u^2 = 2pk$, we find $k = \frac{ds^2\left(Py - \frac{Fgx}{y}\right)}{Pds^2 + Fdx^2}$.

We may remark that $\frac{udx}{ds}$ is the velocity of the body F, for the string being inextensible, the velocity of the body F at each instant is to the velocity of the body P, as dx to ds.

Cor. 1. If we make $g = p$, we have $k = \frac{ds^2 . (Py - Fx)}{Pds^2 + Fdx^2}$, which agrees with the formula given without demonstration, by M. Bernoulli, tome 2, des Memoires de Petersbourg, and may be easily drawn from the principle of the Conservation des Forces Vives.

Cor. 2. If p and $g = 0$, that is to say if the bodies are void of gravity, we have $\frac{u^2 . (Pds^2 + Fdx^2)}{Pds^2} =$ a constant quantity.

Cor. 3. The preceding problem may be resolved with the same facility when the bodies move in a medium, the resistance of which is proportional to any function of the velocity.

For putting $\frac{pdy}{ds} - \varphi u$ in place of $\frac{pdy}{ds}$ (the quantity φu expressing in general a function of u), and $g + \varphi\left(\frac{udx}{ds}\right)$ in place of g, in the equation of the problem, we shall have an equation in which the indeterminate quantities may be separated in several cases, as when $\varphi u = a + buu$, a and b being any constant quantities whatever. Seeing, indeed, that when we know the place of the body F, we know also that of the body P, we have for any situation of the bodies F, P the relation of dx to ds; also from the nature of the curve C P p we know the relation of dy to dx. Lastly, if we make $dx = qds$, we have $ddx = dqds$, and from this we may find the relation of dq to ds. Therefore, supposing $dy = rds$, and $dq = zds$, we have in the present case $2udu\left(\frac{Fqq}{P} + 1\right) + \frac{2Fuuqzds}{P} = 2ds\ (pr - a - bu^2) - \frac{2Fqds}{P} \times (g + a + bu^2q^2)$; an equation in which the indeterminates may be separated by known methods, granting the quadrature of curves.

Cor. 4. If each of the bodies were to move in a curve, then calling (fig. 12, pl. A.) Ff, dt; Fu, dx: and FN, dz, we shall find $\varphi w = \frac{\alpha dt}{ds}$; $w_v = \frac{gdz}{dt} . \frac{ds^2}{u^2}$; and if in place of FN we put $\frac{Fndt}{dx}$, and in place of $- ddx = \frac{\alpha dx}{ds} + \frac{gds^2}{u^2} - n$ we write $- ddt = \frac{\alpha dt}{ds} + \frac{gdz}{dt} . \frac{ds^2}{u^2} - n$; the problem is resolved, and we find

$$d\left(\frac{u^2(Pds^2 + Fdt^2)}{Pds^2}\right) = 2pdy - \frac{2Fgdz}{P};$$

an equation which we might also have found by means of the principle of the Conservation des Forces Vives.

Hence if the primitive value of u be zero, and that then $y = A$ and $z = 0$, we have

$$\frac{u^2(Pds^2 + Fdt^2)}{Pds^2} = 2p \times (y - A) - \frac{2Fgz}{P}.$$

Cor. 5. If in the preceding Cor. we make $u^2 = 2py$, and $p = g$, we have $k = \frac{(P(y - A) - Fz)ds^2}{Pds^2 + Fdt^2}$.

We may, if we please, make $A = 0$, by supposing that the body P departs from C, and then we shall have

$$k = \frac{P(y - Fz)ds^2}{Pds^2 + Fdt^2}.$$

Scholium. This solution of the problem may appear rather long; but I thought it was proper to shew the application of our general principle.

If we would otherwise resolve the problem, we might proceed in this manner.

Let T be the tension of the string, which acts equally according to CP and CF, then we have $\frac{Ppdy}{ds} - \frac{Tdx}{ds}$ for the force which accelerates the body P in the direction Pp, and $\frac{Tdx}{ds} - \frac{Fgdz}{dt}$ for that which accelerates the body F in the direction Ff; we have therefore $\left(\frac{Ppdy}{ds} - \frac{Tdx}{ds}\right) ds = Pudu$, and $\left(\frac{Tdx}{dt} - \frac{Fgdz}{dt}\right) dt = \frac{F}{2} d\left(\frac{u^2dt^2}{ds}\right)$, from whence we get by adding the two equations, and integrating,

$$u^2 + \frac{u^2Fdt^2}{Pds^2} = 2py - \frac{2Fgz}{P}.$$

This last solution is certainly more simple than the preceding one; but then, on the other hand I think it is neither so luminous nor so direct. For strictly speaking, the string does not act on the bodies; it has only a force of resistance and not of impulsion.

ARTICLE VI.

On the Expansion of the Formula $\varphi\,(A + c \cos x)^m$.

By Mr. Thomas Knight, *Papcastle.*

Though mathematicians have been very much occupied with the expansion of $(A + c \cos x)^m$, into the form $B + B_1 \cos x + B_2 \cos 2x + \dots\dots$ (1), from its connexion with the enquiries of Physical Astronomy, no one has shewn *a priori* how B_1 may be found from B. This, besides exhibiting other remarkable re-

markable relations of the coefficients, I propose to do for the general form φ (A + c cos x), where φ represents any function whatever. If we denote generally by $\overset{1}{B}_{n}$ the fluxional co-efficient of B_{n} with respect to A, we shall have

$$\left(\frac{\dot{\overline{\varphi(A+c\cos x)}}}{\dot{A}}\right)^{n} = \overset{1}{B} + \overset{1}{B}_{1}\cos x + \overset{1}{B}_{2}\cos 2x + \ldots + \overset{1}{B}_{n}\cos nx + .$$

Now whether we multiply this by $-c\sin x\dot{x}$ and take the fluent with respect to x, or by $\cos x\,\dot{c}$ and take the fluent with respect to c, we shall in both cases get a series equal to $\varphi(A+c\cos x)$. The first mode of proceeding gives

$$\varphi(A+c\cos x) = -\int\left\{c\overset{1}{B}\sin x\dot{x} + c\overset{1}{B}_{1}\sin x\cos x\dot{x} + \ldots + c\overset{1}{B}_{n}\sin x\cos nx\dot{x} + \right\}$$

$$= -\int\left\{\begin{array}{l} c\overset{1}{B}\sin x\dot{x} + \frac{c}{2}\overset{1}{B}_{1}\sin 2x\dot{x} + \frac{c}{2}\overset{1}{B}_{2}\sin 3x\dot{x} + \ldots + \frac{c}{2}\overset{1}{B}_{n-1}\sin nx\dot{x} + \\ -\frac{c}{2}\overset{1}{B}_{2}\sin x\dot{x} - \frac{c}{2}\overset{1}{B}_{3}\sin 2x\dot{x} - \frac{c}{2}\overset{1}{B}_{4}\sin 3x\dot{x} - \ldots - \frac{c}{2}\overset{1}{B}_{n+1}\sin nx\dot{x} - \end{array}\right\}$$

$$= \text{Const}: + \left(c\overset{1}{B} - \frac{c}{2}\overset{1}{B}_{2}\right)\cos x + \tfrac{1}{2}\left(\frac{c}{2}\overset{1}{B}_{1} - \frac{c}{2}\overset{1}{B}_{3}\right)\cos 2x + . + \frac{1}{n}\left(\frac{c}{2}\overset{1}{B}_{n-1} - \frac{c}{2}\overset{1}{B}_{n+1}\right)\cos nx +$$

which, being compared with the form marked (1), gives

$$B_{1} = c\overset{1}{B} - \frac{c}{2}\overset{1}{B}_{2};\ B_{2} = \tfrac{1}{4}\left(c\overset{1}{B}_{1} - c\overset{1}{B}_{3}\right);\ \ldots\ldots B_{n} = \frac{1}{2n}\left(c\overset{1}{B}_{n-1} - c\overset{1}{B}_{n+1}\right);$$

where n is greater than one. Hence we easily find

$$B_{2} = 2B - \frac{2}{c}\int B_{1}\dot{A};\ B_{3} = B_{1} - \frac{4}{c}\int B_{2}\dot{A};\ \ldots\ldots B_{n} = B_{n-2} - \frac{2(n-1)}{c}\int B_{n-1}\dot{A};$$

which is *one ſolution* of the expansion when we have B and B_{1}.

If we had proceeded according to the other method I pointed out we should have had $\varphi(A + c\cos x)$

$$= \int\left\{\cos x\overset{1}{B}\dot{c} + \cos x\cos x\overset{1}{B}_{1}\dot{c} + \cos x\cos 2x\overset{1}{B}_{2}\dot{c} + \ldots\ldots \cos x\cos nx\overset{1}{B}_{n}\dot{c} + \right\}$$

$$= \int\left\{\begin{array}{l} \cos x\overset{1}{B}\dot{c} + \tfrac{1}{2}\cos 2x\overset{1}{B}_{1}\dot{c} + \ldots\ldots = \tfrac{1}{2}\cos nx\overset{1}{B}_{n-1}\dot{c} + \\ + \tfrac{1}{2}\cos x\overset{1}{B}_{2}\dot{c} + \tfrac{1}{2}\cos 2x\overset{1}{B}_{3}\dot{c} + \ldots\ldots + \tfrac{1}{2}\cos nx\overset{1}{B}_{n+1}\dot{c} + \end{array}\right\}$$

and, by proceding as in the former case, a *second solution* is

$$B_{2} = 2\int\left(\frac{\dot{B}_{1}}{\dot{c}}\right)\dot{A} - 2B;\ B_{3} = 2\int\left(\frac{\dot{B}_{2}}{\dot{c}}\right)\dot{A} - B_{1};\ \ldots\ldots B_{n} = 2\int\left(\frac{\dot{B}_{n-1}}{\dot{c}}\right)\dot{A} - B_{n-2};$$

and by adding these two solutions we get *a third*

$$\underset{2}{B} = \int\left(\frac{\dot{\underset{1}{B}}}{\dot{c}}\right)\dot{A} - \frac{1}{c}\int \underset{1}{B}\dot{A}\,;\ \underset{3}{B} = \int\left(\frac{\dot{\underset{2}{B}}}{\dot{c}}\right)\dot{A} - \frac{2}{c}\int \underset{2}{B}\dot{A}\,;\,..\,\underset{n}{B} = \int\left(\frac{\dot{\underset{n-1}{B}}}{\dot{c}}\right)\dot{A} - \frac{n-1}{c}\int \underset{n-1}{B}\dot{A}\,;$$

which however is more curious than useful, the first solution being much the simplest.

By equating the values of $\underset{2}{B}$, given by the two first solutions, there arises the linear equation

$\dot{\underset{1}{B}} + \underset{1}{B}\frac{\dot{c}}{c} = 2\overset{1}{B}c$, whence $\underset{1}{B} = \frac{2}{c}\int \overset{1}{B}c\dot{c} + \frac{\text{const}}{c}$, where the last term must evidently be omitted, as c cannot be in the denominator. The relation betwixt $\underset{1}{B}$ and B is expressed then by the equation $\underset{1}{B} = \frac{2}{c}\int \overset{1}{B}c\dot{c}$.

If we expand $\varphi\,(A+c\,\cos x)$ by Taylor's theorem, and put for $\cos^2 x$, $\cos^3 x$, &c. their values in cosines of multiple arcs, we find

$$B = \varphi(A) + \frac{\overset{11}{\varphi}\,(A)}{2} \times \tfrac{1}{2}c^2 + \frac{\overset{1111}{\varphi}\,(A)}{2.3.4} \times \tfrac{3}{8}c^4 + \frac{\overset{1\ldots6}{\varphi}\,(A)}{2.3\ldots\ldots6} \times \tfrac{5}{16}c^6 +$$

where by $\overset{1\ldots\ldots n}{\varphi}\,(A)$ I denote the fluxional co-efficient of the nth order of $\varphi\,(A)$.

The equation $\underset{1}{B} = \frac{2}{c}\int \overset{1}{B}c\dot{c}$ gives

$$\underset{1}{B} = \overset{1}{\varphi}(A)c + \frac{\overset{111}{\varphi}\,(A)}{2.3} \times \tfrac{3}{4}c^3 + \frac{\overset{1..5}{\varphi}\,(A)}{2.3.4.5} \times \tfrac{5}{8}c^5 + \frac{\overset{1..7}{\varphi}\,(A)}{2.3\ldots\ldots7} \times \tfrac{35}{64}c^7 +$$

and from these the other co-efficients may be found most easily by the first solution.

ARTICLE VII.

On the Sine and Cosine of the Multiple Arc.

By Mr. KNIGHT.

The demonstrations which mathematicians have given of the formulas for the sine and cosine of the multiple arc, in terms of the cosine of the simple arc, appear to be deficient in brevity and simplicity; they have indeed been established by means of theorems more abstruse than the propositions themselves that were to be investigated. Thus *Waring's* proof is made to depend on his theorem for the sum of the powers of the roots of an equation. *Arbogast*, "Calc. des Derivations" p. 179, uses, for the same purpose, his form for an expanded multinomial. Others have employed the method of increments, see *La Croix*, "Calc. Diff." tom. 3, p. 207. Mr. *Woodhouse* (Trigonometry, p. 174) seems to entertain the opinion that these expressions "cannot be rigorously established by any simple means or obvious artifices". I venture to offer the following method of investigation;—

It is easy to see from the trigonometrical equations

$$\sin nx = 2 \cos x \sin (n-1) x - \sin (n-2) x \ldots\ldots\ldots (1)$$
$$\cos nx = 2 \cos x \cos (n-1) x - \cos (n-2) x$$

that the expressions we are in search of will have the following forms, when n is a whole positive number,

$$\sin nx = \sin x \, (\text{A} \cos x^{n-1} + \text{B} \cos x^{n-3} + \text{C} \cos x^{n-5} +)$$
$$\cos nx = \alpha \cos x^{n} + \beta \cos x^{n-2} + \gamma \cos x^{n-4} +$$

by taking the fluxions of which we get

$$\cos nx = \cos x (\frac{\text{A}}{n} \cos x^{n-1} + \frac{\text{B}}{n} \cos x^{n-3} + \frac{\text{C}}{n} \cos x^{n-5} +)$$
$$+ (\cos x^{2} - 1)(\frac{n-1}{n} \text{A} \cos x^{n-2} + \frac{n-3}{n} \text{B} \cos x^{n-4} + \frac{n-5}{n} \text{C} \cos x^{n-6} +)$$
$$\sin nx = \sin x (\alpha \cos x^{n-1} + \frac{n-2}{n} \beta \cos x^{n-5} + \frac{n-4}{n} \gamma \cos x^{n-5} +)$$

a comparison of the extreme equations give

$$\alpha = \text{A}; \ \beta = \frac{n}{n-2} \text{B}; \ \gamma = \frac{n}{n-4} \text{C}; \ \delta = \frac{n}{n-6} \text{D}; \ \&c \ldots\ldots (2)$$

the middle equations give

$$\beta = \frac{n-2}{n} \text{B} - \frac{n-1}{n} \text{A}; \ \gamma = \frac{n-4}{n} \text{C} - \frac{n-3}{n} \text{B} : \delta = \frac{n-6}{n} \text{D} - \frac{n-5}{n} \text{C}, \&c.$$

and putting for β, γ, δ, &c. their values given by equations (2) we find

$$\text{B} = \frac{-(n-2)}{4} \text{A}; \ \text{C} = \frac{-(n-3)(n-4)}{8(n-2)} \text{B} = \frac{(n-3)(n-4)}{4 \cdot 8} \text{A};$$
$$\text{D} = \frac{-(n-5)(n-6)}{12(n-3)} \text{C} = \frac{-(n-4)(n-5)(n-6)}{4 \cdot 8 \cdot 12} \text{A}; \ \&c.$$

whence equations (2) give

$$\alpha = \text{A}; \ \beta = \frac{-n}{4} \text{A}; \ \gamma = \frac{n(n-3)}{4 \cdot 8} \text{A}; \ \delta = \frac{-n(n-4)(n-5)}{4 \cdot 8 \cdot 12} \text{A}; \ \&c.$$

all the co-efficients are, therefore, known if we know A; and if we form two or three of the first values of $\sin nx$ by the trigonometrical equation (1) it will be immediately apparent, from the mode of formation, that $\text{A} = 2^{n-1}$. We have then

$$\sin nx = \sin x \left\{ (2\cos x)^{n-1} - \frac{n-2}{1}(2\cos x)^{n-3} + \frac{(n-3)(n-4)}{1 \, . \, 2}(2\cos x)^{n-5} + \right\},$$

$$\cos nx = \tfrac{1}{2} \left\{ (2\cos x)^{n} - \frac{n}{1}(2\cos x)^{n-2} + \frac{n(n-3)}{1 \, . \, 2}(2\cos x)^{n-4} + \right\},$$

I will add here what appears to be a very simple method of finding the sine and cosine of an arc, in terms of the arc itself: it does not require the use of fluxions.

With respect to the form of series to be assumed, there is no difficulty in perceiving that we may make

$$\sin x = \overset{(1)}{s}\, x + \overset{(3)}{s}\, x^3 + \overset{(5)}{s}\, x^5 + \; ; \; \cos x = 1 + \overset{(2)}{c}\, x^2 + \overset{(4)}{c}\, x^4 +$$

If we substitute these expressions in the common formula for $\cos(x-u)$, (viz. $\cos(x-u) = \cos x \cos u + \sin x \sin u$) but restrict the substitution, in the second member, to $\sin u$, $\cos u$, there arises

$$1 + \overset{(2)}{c}\,(x-u)^2 + \overset{(4)}{c}\,(x-u)^4 + = \cos x\,(1 + \overset{(2)}{c}\,u^2 +) + \sin x\,(\overset{(1)}{s}\,u +)$$

a comparison of the coefficients* of u and of u^2 gives the two equations

$$\sin x = -2\,\frac{\overset{(2)}{c}}{\overset{(1)}{s}}\,x - 4\,\frac{\overset{(4)}{c}}{\overset{(1)}{s}}\,x^3 - \ldots\ldots - n\,\frac{\overset{(n)}{c}}{\overset{(1)}{s}}\,x^{n-1} -$$

$$\cos x = 1 + 6\,\frac{\overset{(4)}{c}}{\overset{(2)}{c}}\,x^2 + 15\,\frac{\overset{(6)}{c}}{\overset{(2)}{c}}\,x^4 + \ldots\ldots + \frac{n(n-1)}{2}\,\frac{\overset{(n)}{c}}{\overset{(2)}{c}}\,x^{n-2} +$$

which, compared with the forms that were assumed, will give all the co-efficients when we know $\overset{(1)}{s}$; concerning the finding of which I have nothing new to offer.

This method may be compared with that of *La Croix*, in the introduction to his Calc. Diff. which appears somewhat awkward. His expansion of the exponential might also have been neater.

Let $a^x = 1 + A'x + A''x^2 + \ldots. + \overset{(n)}{A}x^n + \ldots\ldots.., \quad (\alpha)$

then $a^{x+u} = a^x(1 + A'u + A''u^2 +)$ or $= 1 + A'(x+u) + A'(x+u)^2 +$ and by comparing the co-efficients of u, in these two values,

we have $a^x = 1 + 2\,\frac{A''}{A'}\,x + 3\,\frac{A'''}{A'}\,x^2 + \ldots. + n\,\frac{\overset{(n)}{A}}{A'}\,x^{n-1} +$

and this compared with (α) gives all the co-efficients by means of the first. I shall terminate this miscellaneous paper with a remarkable expression for the elliptic quadrant, which I do not remember to have seen noticed before. Let e', e'', e'', &c. be the same as in Mr. *Wallace*'s formula; also

---

* I suppose the binomials in the first member to be expanded as far as u^2.

$$P=(1+e')(1+e'')\,..\,(1+\overset{(n)}{e});\; P'=\frac{P}{1+e'};\; P''=\frac{P'}{1+e''};\; P'''=\frac{P''}{1+e'''};\text{\&c.}$$

$$R=2n-(1-e')PP'-2(1-e'')P'P''-2^2(1-e''')P''P'''-2^3(1-e'''')P'''P''''-\text{\&c.}$$

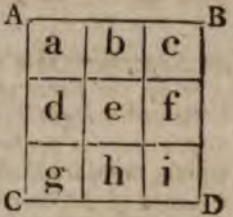
The number of terms taken of R must be $n+1$; and n will be 2, 3, 4, &c. according to the degree of approximation required.

Then the elliptic quadrant $=\frac{\pi}{2}\times\frac{R}{P}$.

ARTICLE VIII.

Solution to the Problem of making a Magic Square of nine Cells.

By Mr. NOBLE *of the Royal Military College.*



Let ABCD be the square required, omitting for the present the condition of the numbers making an arithmetical progression, and retaining only that of the horizontal, perpendicular and diagonal bands, making the same sum; which sum denote by s. Then the conditions of the problem give the following equations, $a+b+c=c+f+i=g+d+a=a+e+i=c+e+g=b+e+h=d+e+f=s$; this last equation, viz. $s=d+e+f$ is not independent of the rest but deducible from them, thus: $3s=c+f+i+g+d+a+b+e+h=d+e+f+a+b+c+i+g+h=d+e+f+2s$; therefore $s=d+e+f$; it is evident any other of the sums might have been taken instead of $d+e+f$; therefore the problem furnishes only six independent conditions.

Now $a+e+i+c+e+g+d+e+f=3e+a+d+g+c+f+i$, that is $3s=3e+2s$, or $s=3e$.

Hence $b+h=d+f=g+c=a+i=2e$.

Put $a-i=2D$, $b-h=2\Delta$, then $a=e+D$, $i=e-D$, $b=e+\Delta$, $h=e-\Delta$,

$a+b+c=3e$, therefore $c=3e-(a+b)=e-D-\Delta$

$g+c=2e$, therefore $g=2e-c=e+D+\Delta$

$c+f+i=3e$, theref $f=3e-(c+i)=3e-(e-D-\Delta+e-D)=e+2D+\Delta$

$d+f=2e$, therefore $d=2e-f=e-2D-\Delta$. These values may be thus arranged,

$$
\begin{aligned}
d &= e - 2D - \Delta \\
c &= e - D - \Delta \\
h &= e - \Delta \\
i &= e - D \\
e &= \frac{s}{3} \\
a &= e + D \\
b &= e + \Delta \\
g &= e + D + \Delta \\
f &= e + 2D + \Delta
\end{aligned}
$$

and will constitute an arithmetical progression by making $\Delta = 2D$; which solves the problem in the utmost generality for the given number of cells, viz. 9.

If $D = 1$ and $\frac{s}{3} = 5$, the square will consist of the nine digits.

Almost exactly in the same way we may proceed when products are substituted for sums and a geometrical for an arithmetical progression.

I am not aware of the existence of any investigation independent of tentative methods; such, for instance, as that in Ronayne's Algebra, which is the only I have ever seen.

ARTICLE IX.

An Experiment that might be made, in order to discover whether, or no, the Force of Gravity, by which Bodies fall towards the Earth in right Lines perpendicular to its Surface, and which, if it were a perfect Sphere, would tend to its Centre, arises from the mutual Attraction of all the Particles of Matter contained in it towards each other, as Sir Isaac Newton *supposes.*

By FRANCIS MASERES, *Esq. Cursitor Baron of the Exchequer.*

Sir Isaac Newton supposes that every particle of matter in the earth, and in the moon, and in the sun, and in all the planets and their satellites, and in all the comets, that, from time to time, appear in the solar system, attracts every other particle of matter with a force that decreases in the same proportion as the square of the distance between the two particles increases; excepting the cases in which the distance between the two particles is extremely small, (as for example, less than the millionth part of an inch,) in which cases he supposes a contrary principle of *repulsion* to take place between the particles. And upon these suppositions he concludes that, when a body is situated within the earth, or below its surface, its gravity, or tendency towards the centre of the earth, will be less than it would be on the surface of the earth; because it will be attracted upwards by the super-incumbent mass of earth, as well as downwards by the still greater mass of earth below it: and he demonstrates that, in consequence of these contrary attractions, its gravity, or tendency towards the centre of the earth, will grow continually weaker and weaker as it approaches to that centre, and will totally cease when it has fallen to that centre: and he further demonstrates that, if the whole globe of the earth, consisted of matter that was every where of the same density, from its surface to its centre, (which, however, he does not suppose to be the case) the force of gravity wherewith any body would be attracted towards the centre of the earth, at different distances from the centre, would be always proportional to those distances.

This doctrine of the mutual attraction of the particles of matter Mr. Christian Huyghens, the great Mathematician and Philosopher of Holland, in the 17th century, could never assent to, though he lived to the year 1695, which was eight years after the publication of Sir Isaac Newton's famous book, intitled "*Philosophiæ Naturalis Principia Mathematica.*" He was one of the very few persons (probably not more than a dozen in all Europe) who were capable of reading and understanding that very learned and difficult work; and he did read it with great attention and admiration; as will appear from the following passages of his discourse on the cause of gravity, or the weight of bodies, which was published, together with his Treatise on Light, (both in French) in a small quarto volume, at Leyden in Holland, in the year 1690. In page 159 of that volume, he speaks of that celebrated work in these words:—"Et du depuis, ayant encore "lû le *tres scavant Ouvrage* de Monsieur Newton, dont le titre "est '*Philosophiæ Naturalis Principia Mathematica*:'" and, afterwards, in page 165, he expresses himself in these words:—"J'ai crû devoir aller au-devant de ces objections que pourroit "suggérer le livre de Monsieur Newton, sachant la grande "estime qu'on fait de cet Ouvrage, et avec Raison; puisqu'on "ne sauroit rien voir de plus savant en ces Matiéres, ni qui "temoigne une plus grande Pénétration d'Esprit." And in this same discourse he approves highly of several of the philosophical discoveries of Newton set-forth in that great work; as, that the force of gravity by which the moon tends towards the earth, and is retained in her orbit, is to the force of gravity of a body at the surface of the earth inversely as the squares of their respective distances from the earth's centre; and that the gravitation of the primary planets towards the sun observes the same law of the inverse ratio of the squares of their distances from the sun, which is the central body towards which they tend: and he declares that he thinks that Mr. Newton has fully refuted Monsieur Des Cartes's opinion that the planets are carried round the sun by vortexes; and he agrees with Mr. Newton in some other important philosophical doctrines contained in that celebrated treatise. But yet he declares that he cannot assent to Mr. Newton's doctrine of the mutual attraction of the particles of matter. This he expresses in these words, in page 159:—“ Je ne suis pas d'accord [avec Monsieur Newton] d'un principe qu'il suppose dans ce calcul et alleurs: qui est, que toutes les peties parties qu'on peut imaginer dans deux, ou plusieurs, différens corps, s'attirent, ou tendent à s'approcher, mutuellement. Ce que je ne saurois admettre, parceque je crois voir clairement que la cause d'une telle attraction n'est point explicable par aucun principe de méchanique, ni des régles du mouvement. Comme je ne suis persuadé non-plus de la nécessité de l'attraction mutuelle des corps entiers: ayant fait voir que quand il n'y auroit point de terre, les corps ne laisseroient pas, par ce qu'on appelle leur pesanteur, de tendre vers un centre.” And again, in page 163, he has these words:—“ Ce seroit autre chose si on supposoit que la pesanteur fût une qualité inhérente de la matiere corporelle. Mais c'est a quoi je ne crois pas que Monsieur Newton consente, parce q'une telle Hypothése nous éloigneroit fort des Principes Mathématiques ou Méchaniques.”

These are the passages in that discourse of Mr. Huyghens, which relate to this doctrine of the mutual attraction of the particles of matter in all directions whatsoever, as well as in right lines tending to the centre of the earth. But now this doctrine seems to be generally admitted by natural philosophers, both in in Great Britain and on the Continent.

It is, however, much to be wished that some positive proofs of the existence of such a principle of mutual attraction of the particles of matter could be given, that were derived from actual experience, that we might assent to this doctrine of the general attraction of all particles of matter-towards each other, in all directions whatsoever, as well as in the direction of right lines tending to the centre of the earth, upon the same grounds as we assent to the doctrine of a mutual attraction between a loadstone and a piece of iron, when we see the piece of iron actually move towards the loadstone; and when, if the piece of iron is held fast, and prevented from moving towards the loadstone, we see the loadstone move towards the piece of iron.

But, if this mutual attraction of all the particles of matter towards each other, in all directions and at all distances, does really exist in Nature, (as is now generally admitted by philosophers) it is extremely difficult to contrive experiments that will render the effects of it perceptible by our senses. Yet some experiments have been made with this view, by very eminent philosophers, which are thought by good judges to have been attended with considerable success; for Dr. Nevil Maskelyne, the present Astronomer Royal, took a journey into Scotland for this purpose, about thirty years ago, and made some very nice observations on the positions of some bright fixt stars situated nearly in the zenith of the heavens, first on the south side, and then on the north side, of a very high and broad mountain, called *Schehallien*, and found a difference in the zenith of the said stars on the two different sides of the said mountain, which, he thought, afforded a proof that the plumb-line, employed in measuring those zenith distances, had been drawn a little out of its proper direction on the opposite sides of the mountain, by its attraction. And a like conclusion in favour of this doctrine of the attraction of matter is said to have been drawn by the French Astronomer, Mons. de la Condanime and his companions, (who were sent, in the year 1744, by Lewis the XVth, King of France, to measure a degree of latitude in South America, from observations (similar to those of Dr. Maskelyne abovementioned) which they had made on different sides of the high and broad range of mountains called *The Cordilleras*. And a few years ago, the very learned and ingenious Natural Philosopher, Mr. Henry Cavendish, made an experiment of a different kind, but for the same purpose, in England, of which an account has been published in the Philosophical Transactions. And this experiment is likewise considered as a successful one, and as furnishing an additional proof of the attractive power of matter. It is, however, still desirable that we should obtain other proofs of it from experiments properly contrived for the purpose; and I am inclined to think that the following experiment might afford us some further grounds for supposing that this remarkable property of attraction does really belong to matter.

I have been told that in the north of England there are coal mines that are of the depth of six hundred feet, in a perpendicular direction below the surface of the earth; therefore, if the above doctrine of the mutual attraction of the particles of matter is true, the weight of a body carried to the bottom of such a coal mine, would be less than its weight when placed on the surface of the earth; and, if the earth be every where of the same density from the surface to the centre, the weight of the body in the coal mine will be less than its weight on the surface of the earth, in proportion of its distance from the centre of the earth, when it is in the coal mine, to its distance from the centre when it is on surface of the earth; that is, in the proportion of the semi-diameter of the earth — 600 feet, to the semi-diameter of the earth; or, (if we suppose the semi-diameter of the earth to be equal to 4000 miles), of 4000 miles — 600 feet to 4000 miles; but a mile is = 5280 feet, and, consequently 4000 miles are = 4000 times 5280 feet, or 21,120,000 feet, and 4000 miles — 600 feet are (= 21,120,000 feet — 600 feet) = 21,119,400 feet;

therefore the weight of a body in the coal mine will be to its weight on the surface of the earth as 21,119,400 feet are to 21,120,000 feet, or as 211,194 is to 211,200; or the force of gravity acting upon a body in the coal mine, and tending to move it in a right line towards the centre of the earth, will be to the force of gravity acting upon the same body on the surface of the earth, and tending to make it move in the same direction, or in a right line towards the centre of the earth, is as 211,194 is to 211,200. And, consequently, if in any small portion of time a body placed on the surface of the earth would fall perpendicularly downwards through any space called S, it would, if placed in the coal mine, fall perpendicularly downwards through only $\frac{211,194}{211,200}$ parts of the said space S in the same time.

Therefore if a very good pendulum clock, that swings seconds, or performs one oscillation in a second of time, were to be carried to the bottom of such coal mine 600 feet deep, the space through which a heavy body would fall in the said coal mine, in the same time in which it would fall, when on the surface of the earth, through the length of the said pendulum (which we will call L) will be less than the length of the said pendulum, or than L, in the proportion of 211,194 to 211,200, or will be equal to $\frac{211,194}{211,200} \times L$, or $0·999,971,590,909 \times L$, and consequently the time of falling through the whole length of the pendulum L in the coal mine, will be longer than the time of falling through it on the surface of the earth, in the same proportion as it is longer than the time of falling through $0·999,971,590,909 \times L$, in the coal mine.

But the time of falling through the whole length L in the coal mine is to the time of falling through $0·999,971,590,909 \times L$ in the coal mine in the sub-duplicate ratio of the spaces fallen through, to wit L, or 1 L, and $0·999,971,590,909 \times L$, that is in the sub-duplicate ratio of 1 to 0·999,971,590,909, or in the ratio of 1 to the square root of 0·999,971,590,909, or of 1 to 0·999,985,795, or very nearly of 1 to 0·999,985,8.

Therefore the time of falling through L, or the length of the pendulum, in the coal mine, will be to the time of falling through the same length on the surface of the earth, (which is equal to the time of falling through $0.999,971,590,909 \times L$ in the coal mine,) as 1 is to 0.999,985,8.

But the time of falling through L, or the length of the pendulum, in the coal mine, will be to the time of performing one oscillation of it in the coal mine in the same proportion as the time of falling through the length of the pendulum on the surface of the earth is to the time of performing one oscillation of it on the surface of the earth; therefore, *permutando*, the time of falling through the length of the pendulum, in the coal mine, will be to the time of falling through the same length on the surface of the earth, in the same proportion as the time of performing one oscillation in the coal mine is to the time of performing one oscillation on the surface of the earth; but the time of falling through the length of the pendulum, in the coal mine, is to the time of falling through the same length on the surface of the earth as 1 is to 0·999,985,8; or as 10,000,000 is to 9,999,858; therefore while the pendulum would perform 10,000,000 oscillations on the surface of the earth, it would, when placed in the coal mine, perform only 9,999,858 oscillations: but the pendulum is supposed to swing seconds, or to perform one oscillation in a second of time, when on the surface of the earth; and consequently in 10,000,000 seconds of time it will perform 10,000,000 oscillations. Therefore if the clock is placed in the coal mine its pendulum will, in the course of 10,000,000 seconds of time, perform only 9,999,858 oscillations, or 10,000,000 — 142 oscillations, or 142 fewer oscillations than it would have performed in the same time on the surface of the earth; and, therefore, the time exhibited by the clock, when placed in the coal mine, will, at the end of 10,000,000 seconds of time be behind the true time of 10,000,000 seconds (that would have been exhibited by it on the surface of the earth) by 142, or in other words the clock in the coal mine will, in the course of 10,000,000 seconds of time, lose 142 seconds.

But there are 60 seconds in a minute, and 60 minutes in an hour, and 24 hours in a day. Therefore a day will contain 60 × 60 × 24 seconds, or 3600 × 24 seconds, or 86,400 seconds. Therefore 10,000,000 seconds will be equal to $\frac{10,000,000}{86,400}$ days, or 115·74 days, or nearly 116 days.

Therefore in 116 days the clock in the coal mine will lose 142 seconds, and consequently in one day it will lose the 116th part of 142 seconds, that is $\frac{142}{116}''$, or $1''\cdot 224$, or 1 second and more than $\frac{2}{10}$, or $\frac{1}{5}$ of a second.

Therefore if the clock, when carried to the bottom of the coal mine, and left there for two or three months together, should be found at the end of such a time to have lost about a second a day, and this experiment, being often repeated with different clocks, should always give the same result, it would afford us a just ground for concluding that such loss of time must be owing to a diminution of the force of gravity that moves the pendulum, occasioned by its removal from the surface of the earth to the bottom of the coal mine, and that this diminution of the gravity will have arisen from the attraction upwards of the superincumbent mass of matter, which will have weakened its attraction downwards by the much greater mass of matter lying below it.

This computation of the time that would be lost by the pendulum clock in the coal mine, in the space of 116 days, is founded on a supposition, that the matter of the earth was of the same density in all its parts from its surface to its centre. But this supposition is thought to be far from the truth; and it is generally thought that the density of the earth near its centre is much greater than near its surface; and therefore the loss of a body's gravity by its being carried into a coal mine six hundred feet deep, will be much less than on the foregoing supposition, and consequently the loss of time of the pendulum clock in a coal mine of that depth, in the space of 116 days, will be much less than on the former supposition, or than 142 seconds, and, perhaps, not above a fourth part of that time, or 35 seconds and a half; or, neglecting the fraction, 35 seconds. Therefore, there is no reason to expect a greater loss of time from this removal of the pendulum clock to the coal mine than that of 35, or perhaps 30 seconds in 116 days, or $\frac{30}{116}$ seconds, or $\frac{258}{1000}$, or about a quarter of a second in 1 day, or a second in 4 days, or a minute in 240 days, or in 8 months. This is indeed a very small quantity, and renders the experiment more difficult and precarious than it appeared before. Yet, perhaps, it may be worth trying.

* * *

ARTICLE X.

Remarks upon Mr. GOUGH'S *Essay on Polygonal Numbers.*

By Mr. BARLOW.

It was not my intention to enter into any further arguments on the subject of Mr. Gough's demonstration of Fermat's theorem, considering what I had said in Nicholson's Journal to be conclusive, unless it could have been refuted; but instead of taking this course, Mr. Gough has published in your last number a corrected essay, and calls upon me to point out any error; which, he says, in its present form, will be easily detected, if any such should have inadvertently been comitted. It is true, in the corrected essay the error is so obvious, that I trust it only wants to be stated, in order to its being acknowledged, not only by every other mathematician, but by Mr. Gough himself.

As, however, the paper contains several, that appears to be press errors, these ought in candour to be corrected, in order to distinguish them from what I conceive to be an error in principle, and accordingly the following errata should be previously stated:—

Line 10. Introductory letter, after "*logically*," the words, *but not mathematically*, seem to have been omitted.

Line 2, prop. v, for $\frac{a^2-a}{2} \times \frac{a-e}{m-2}$ read $\frac{a^2-a}{2} + \frac{a-e}{m-2}$.

Line 5, of the same proposition, requires the same correction.

Line 5, prop. vii, for $\frac{e}{m-2}$, read $\frac{2e}{m-2}$.

Line 7, prop. viii, for (by cor. 3 prop. ix), read (by cor. 3, prop. vi).

Having thus corrected what I conceive to be press errors, I shall next proceed to state what appears to me to be an error in principle; and which, notwithstanding the formidable display of propositions, corollaries, scholiums, examples, &c. I trust I shall be able to make appear to Mr. Gough's satisfaction, and in very few words, because there is nothing particular to object to till we arrive within 12 lines of the conclusion of the essay; till then, Mr. Gough has only considered the case in which e does not exceed $3m$, and so far his demonstration is just, the only objection I have to it being, that what he has proved in nine propositions, was demonstrable in as many lines. Thus far the demonstration extends, no farther, than the number 9 for the case of triangular numbers, 12 for squares, 15 for pentagonals, &c. and, therefore, the principal point under consideration, viz. when $e > 3m$, is still to prove, and this Mr. Gough dispatches in the following 12 lines:—

" Again, suppose $e > 3m$, then if any one of the values of a found in prop. vii. be an integer, it fulfils the conditions of the proposition. For the first makes e a polygon of the denomination m; the second resolves it into 2, one of which is of the first order; and the third value of a resolves e into $m =$ polygons of the same denomination. Lastly, if all the values of a found by prop. vii. be fractional, and $e > 3m$", (*and which Mr. Gough will observe is the principal thing that requires demonstration*) " the difference of a found in example 2d and 3d, prop. vii. exceeds $m-2$; therefore a value of a, of the form $a = p + r \times (m-2)$, falls between these limits, which resolves e, into m polygons of the same denomination, at most, (by cor. 3, prop. 2".

Now any person who is at all acquainted with mathematical demonstrations, would naturally refer to cor. 3, prop. ii, for some satisfaction on this head, and in a *corrected essay*, at least, would expect to find there something to the purpose, which warranted the above conclusion; this, however, does not appear to be the case, for the corollary in question is as follows;—

"If s, S be aggregates of functions, the sum of whose indices are $\dot{s}$, $\dot{S}$; and if $s > S$, but $s < S$; then the functions into which s is resolved by $\dot{s}$, are not more than those into which S is resolved by $\dot{S}$. For put $r = s - S$, and its index $\dot{r}$ may be found by cor. 4, prop. i. Hence $\dot{S} + \dot{r}$ will resolve s into more functions than $\dot{s}$, by cor. 1.; but $S + r$ resolves $\dot{s}$ into more functions by one than S resolves $\dot{S}$; hence the truth of the corollary;" *and hence also the truth of the proposition.* And here it is that I conceive the error in the demonstration to rest, for I positively deny, that from this corollary any such conclusion can be drawn; and if it be not, as it appears to me to be, entirely unconnected in the principle with the concluding part of the essay, it is at least by far too remote for any mathematical demonstration; and if this latter were the case, would not, and ought not, Mr. Gough, in a corrected essay, where he has been so profuse in propositions and corollaries, to have bestowed a few lines in this place, in order to render evident the most essential part of the demonstration; and as he has not done it am I not justified in concluding, that this omission is not accidental, but proceeds from a defect in the principles of the demonstration itself.

I shall now finally take my leave of this subject. Mr. Gough's first essay, and my objections to it; as also his corrected essay, and my remarks, are before mathematicians; who will, I trust, find no difficulty in deciding between us.

Royal Military Academy,
Woolwich, October 1, 1810.

ARTICLE XI.

On the Resolution of the Irreducible Case in Cubic Equations.

By Mr. P. BARLOW, *of the Royal Military Academy, Woolwich.*

The resolution of cubic equations, particularly those that fall under what is called the *irreducible case*, is a subject which has engaged the attention of many celebrated mathematicians, and we have accordingly several ingenious rules and formulæ for finding, or expressing, the roots of equations of this kind; viz. by trigonometry, whence the roots are obtained by means of table of sines and tangents; by continued fractions, by series, &c. &c. But these methods, though they do honour to the ingenuity and theoretical knowledge of their respective authors, are still very deficient, as they do not lead to the result with that simplicity and ease which is always so much to be desired in mathematical calculations. The trigonometrical formulæ which exhibit the roots of cubic equations are, to the eye, as simple and elegant as can be desired, but the operation (except in a few cases when a particular relation subsists between the co-efficient and the absolute term) is very tedious and operose, and such as to render the solution by that method very dubious beyond 5 or 6 figures; and a similar objection may be made against every other rule that has been at present discovered for the solution of these equations.

The fact is, that those authors, who have most considered this subject, appear to have been too much engaged with the theory, to pay that attention to the practical utility of their deductions that the importance of the subject demanded, and the consequence of this is, as we before observed, that while we are in possession of many elegant rules *in abstracto* for the solution of this problem, we have none that can be applied to real operations with any tolerable degree of ease and convenience.

These circumstances led me to the consideration of the question under a more practical point of view, and the result of my investigation I will endeavour to explain in the following pages; previous to which, however, it will be proper to state a few propositions by way of lemmas, the demonstrations to which are not given, as they may be found in most writers on this subject.

1. Every cubic equation $x^3 + ax^2 + bx + c = 0$ may be transformed to another of the form $y^3 + py = q$.

2. This reduction is performed by taking $-p = 3a^2 - 9b$, and $q = 9ba - 2a^3 - 27c$, in which case $x = \frac{y - a}{3}$.

So that the value of y being determined in the transformed equation, that of x in the original one will be obtained from the equation $x = \frac{y - a}{3}$.

3. Every cubic equation when reduced to the form $y^3 + py = q$, may be resolved by Cardan's rule, if p be positive; and also if p be negative and $4p^3 \angle 27q^2$.

4. But if p be negative, and $4p^3 > 27q^2$, the root, according to the method of Cardan, falls under an imaginary form, and for

this reason it is called the *irreducible case*, the solution of which forms the principal consideration in the present paper.

5. Every equation $x^3 - px = q$, that falls under the irreducible case, has three real roots; the greatest of which is positive when q is positive, (standing on the right hand side of the equation), and negative when q is negative; and in both cases the sum of the two less roots is equal to the greater one, but with the contrary sign; the sum of the three roots being zero, according to the known theory of equations.

These premises being made, I shall proceed to an explanation of the method of solving those cubic equations that are of the imaginary form by means of the following table; and as we have seen that every cubic equation may be reduced to one of the form $y^3 + py = q$, or to the form $y^3 - py = \pm q$ in the case at present under consideration, it will be sufficient for our purpose to limit our investigations to equations of the latter form.

Now the equation $y^3 - py = q$ may be reduced to another dependent equation $y'^3 - y' = q'$, where $q' = \frac{q\sqrt{p}}{p^2}$.

For make $y = \frac{y'}{n}$, then the first equation becomes $\frac{y'^3}{n^3} - \frac{py'}{n} = q$, or $y'^3 - n^2py' = n^3q$, and if now we put $n^2p = 1$, we have $n = \frac{1}{\sqrt{p}}$, or $n^3 = \frac{\sqrt{p}}{p^2}$; and consequently $n^3q = \frac{q\sqrt{p}}{p^2}$, and thus the equation $y^3 - py = q$ is transformed to another $y'^3 - y' = \frac{q\sqrt{p}}{p^2}$, or making $\frac{q\sqrt{p}}{p^2} = q'$ the equation becomes

$$y'^3 - y' = q'.$$

And since $y = \frac{y'}{n}$, and $n = \frac{1}{\sqrt{p}}$, we have $y = y'\sqrt{p}$, and consequently when we have found the value of y' in the equation $y'^3 - y' = q'$, we shall have immediately that of y in the equation $y^3 - py = q$, because $y = y'\sqrt{p}$.

And since the original equation $y^3 - py = q$ is of the irreducible form, the dependent transformed equation $y'^3 - y' = q'$, is so likewise, and consequently, from what has been before observed, $\frac{q'^2}{4} \angle \frac{1}{27}$, or $q'^2 \angle \frac{4}{27}$, or $q' \angle \frac{\sqrt{12}}{9}$ that is $q' \angle$ $\cdot 384900 17$.

And if now this number be substituted as the maximum value of q' in the above equation, making $y'^3 - y' = \cdot 314900179$, we shall find by solving this cubic that the greatest value of y, can never exceed 1·1541, nor the least value of the same root ever be less than 1; that is, every possible value of y' in equations of the form $y'^3 - y' = q'$, falling under the irreducible case, are contained between the limits 1 and 1·1541 : but if the equation assume the form $y'^3 - y' = -q'$, then the limits above given will be those of the negative roots.

Having observed this circumstance with some attention, I was led to the formation of the following table, which exhibits the value of q' for all possible values of y' between the above limits, to five places of figures; and those of q' to eight places; by means of which we may ascertain by inspection the greatest root of all equations of this kind to five places of figures, and by a single division these may be extended accurately to eight or nine places of figures, as will be shewn in the sequel.

The construction of the table is so obvious that very little need be said by way of explanation, it will be sufficient for this purpose to observe, that in the first vertical column in each page are found the four leading figures of y', and the fifth is found in the upper horizontal line over the column in which the value of q' is observed to lie; being the same as the construction of the common tables of logarithms, and the method of finding y' is the same as that for finding the natural numbers to any given logarithm.

Thus for example in the equation

$$y'^3 - y' = \cdot 31063461$$

we find the value of q' in the table opposite 1·129, and in the column marked 2, and therefore, the true value of y' to five places in 1·1292.

Again, let there be proposed the cubic equation $y^3 - 4y = 1$, to find the root to five places.

Here $p = 4$, $q = 1$, and therefore $\frac{q\sqrt{p}}{p^2} = \cdot 125000$, which in the table corresponds to 1·0575, and, consequently, 1·c575 $\times \sqrt{p} = 1\cdot 0575 \times 2 = 2\cdot 1150 = y$ the root required.

Let now, $x^3 - 4x^2 + x + 3 = 0$, be the proposed equation.

First $a = -4$, $b = 1$, and $c = 3$; therefore, $p = 3a^2 - 9b = 39$ and $q = 9ab - 2a^3 - 27c = 11$.

The reduced equation therefore becomes

$$y^3 - 39y = 11$$

Also $$\frac{q\sqrt{p}}{p^2} = \frac{11\sqrt{39}}{39^2} = \cdot 04516429$$

which corresponds in the table to 1·0219,

therefore, $1\cdot 0219 \times \sqrt{39} = 6\cdot 4377 = y$

then $x = \frac{y - a}{3} = \frac{10\cdot 4377}{3} = 3\cdot 47923$ is one of the roots of the proposed equation.

At present we have only taken the root to five places of figures, but it will require very little more trouble to ascertain the same to nine places, (which is sufficiently near for most practical purposes), as may be illustrated in the following manner.

If a and b be made to represent any quantities that are very small with regard to a third quantity y', then the difference between $y'^3 - y'$ and $(y' + a)^3 - (y' + a)$; is to the difference between $(y'^3 - y')$ and $(y' + b)^3 - (y' + b)$, very nearly in the ratio of a to b: for a and b being very small with regard to y', $(y' + a)^3$ and $(y' + b)^3$ will be very nearly expressed by $y'^3 + 3y'^2a$ and $y'^3 + 3y'^2b$; and thus the differences above stated will become $3y'^2a - a : 3y'^2b - b$, which are exactly in the ratio of $a : b$. And hence it appears, that the difference between any two consecutive values of q' in the table, is to the difference between either of them and a third intermediate value of q', as the difference between the two consecutive values of y' is to the difference between one of those values and the required intermediate value of y'.

So that when any given value of y' is not found exactly in the table, we must take out the difference between the next greater value and the next less, as also between the less and that proposed; then, as the first of these differences is to ·0001, the difference between the corresponding values of y', so is the latter difference to the proportional part to be added on to the least tabular value of y': and since the above ratio approaches extremely near to that of equality, this proportional part may be considered as correct to four places of figures, and thus the value of y' will be accurately determined to at least 9 figures.

As an example, let there be proposed the equation $x^3 - 7x = 6$.

First $\frac{6\sqrt{7}}{7^2} = \cdot 32396955$

	value of q'		value of y'
next greater tabular number is	·32398835	....	1·1339
next less	·32370267	....	1·1338
difference......................	28568	....	0001
difference between the less and the given number	26688		

Therefore from the foregoing rule, we have

28568 : ·0001 :: ·26688 : 00009342

which being added to the less tabular numbers gives 1.13389342 for the required value of y', and consequently, $y = 1.13389342 \times \sqrt{7}$.

$$\begin{array}{lr}
\text{See the operation} & 1{\cdot}1338934\;2 \\
 & \underline{3157546{\cdot}2} = \sqrt{7}\ \text{inverted} \\
 & 2\;2677868\;4 \\
 & 6803360\;5 \\
 & 453557\;4 \\
 & 56694\;7 \\
 & 7937\;3 \\
 & 567\;0 \\
 & 11\;3 \\
 & \underline{3\;4} \\
\text{therefore } y = & \underline{3{\cdot}00000000}
\end{array}$$

which is true to the ninth place of figures, the real value of y being 3, as is evident from the given equation.

Hence it appears that the only tedious operations to be performed in solving the irreducible case of cubic equations by this method is, first, extracting the root of the co-efficient, (when it is not an exact square) one operation in division, and the above multiplication, but these are comparatively very trifling when contrasted with any other method, at least that I am acquainted with; and the first of these operations, if the co-efficient be not large, may be supplied from books of tables where the roots are given to a limited extent. It is presumed that the examples which have been given are quite sufficient for the purpose of illustrating the use of the table, and I shall therefore only observe, that the operation is exactly the same in equations of the form $y^3 - py = -q$, but the root so found will be negative, whereas in the former case it was positive.

In the foregoing pages we have only taught the method of finding one of the roots of a given equation, but it is evident that the table might have been so formed as to exhibit all the three roots, which would, however, only have extended it to an unnecessary length; for one of the roots being found, the other two are readily discovered by the following formulæ.

Let the known root be represented by a; p and q being the co-efficient and absolute term as before; then, the two unknown roots b and c will be found thus

$$b = \tfrac{1}{2}a + \tfrac{1}{2}\sqrt{p - \frac{3q}{a}}$$

$$c = \tfrac{1}{2}a - \tfrac{1}{2}\sqrt{p - \frac{3q}{a}}$$

these two roots being negative when a is positive, and positive when a is negative.

Thus in the equation $y^3 - 8y = 3$
where one of the roots is $+ 3 = a$, the other two roots are

$$b = -(\tfrac{3}{2} + \tfrac{1}{2}\sqrt{5})$$

$$\text{and } c = -(\tfrac{3}{2} - \tfrac{1}{2}\sqrt{5}).$$

y′	0	1	2	3	4
1000	00000000	00020003	00040012	00060027	00080048
1001	00200300	00220363	00240432	00260507	00280588
1002	00401201	00421324	00441453	00461588	00487294
1003	00602703	00622886	00643075	00663271	00683472
1004	00804806	00825050	00845299	00865555	00885817
1005	01007513	01027816	01048126	01068442	01088764
1006	01212822	01231186	01251556	01271932	01292314
1007	01414734	01435159	01455589	01476026	01496469
1008	01619251	01639736	01660227	01680724	01701227
1009	01824373	01844918	01865470	01886027	01906591
1010	02030100	02050706	02071318	02091936	02112560
1011	02236433	02257100	02277772	02298451	02319136
1012	02443373	02464100	02484854	02505573	02526319
1013	02650920	02671708	02692501	02713302	02734109
1014	02859074	02879923	02900778	02921639	02942507
1015	03067838	03088747	03109663	03130585	03151518
1016	03277210	03298180	03319157	03340140	03361129
1017	03487191	03508223	03529261	03550305	03571355
1018	03697783	03718876	03739974	03761080	03782190
1019	03908986	03930140	03951300	03972466	03993638
1020	04120800	04142015	04163236	04184464	04205697
1021	04333226	04354502	04375785	04397074	04418368
1022	04546265	04567602	04588946	04610296	04631652
1023	04759917	04781316	04802721	04824132	04845549
1024	04974182	04995643	05017109	05038582	05060061
1025	05189063	05210584	05232112	05253646	05275187
1026	05404558	05426141	05447730	05469326	05490928
1027	05620668	05642313	05663964	05685622	05707285
1028	05837395	05859102	05880815	05902534	05924259
1029	06054739	06076507	06098282	06120062	06141849
1030	06270000	06294530	06316366	06348209	06360057
1031	06491279	06513171	06535069	06556973	06578884
1032	06710477	06732431	06754391	06776357	06798329
1033	06930294	06952309	06974331	06996360	07018394
1034	07150730	07172808	07194892	07216982	07239079
1035	07371788	07393927	07416073	07438226	07460384
1036	07593466	07615668	07637876	97660090	07682311
1037	07815765	07838029	07860300	07882577	07904859
1038	08038687	08061014	08083346	08105688	08128030
1039	08262232	08284631	08307016	08329417	08351824
1040	08486400	08508851	08531308	08553772	08576242
1041	08711192	08733705	08756225	08778751	08801284
1042	08936609	08959185	08981767	09004356	09026951
1043	09162651	09185289	09207934	09230585	09253243
1044	09389318	09412020	09434727	09457441	09480161
1045	09616613	09639376	09662147	09684923	09707706
1046	09844534	09867360	09890193	09913032	09935878
1047	10073082	10095972	10118867	10141769	10164678
1048	10302259	10325211	10348170	10371135	10394106
1049	10532065	10555080	10578101	10601129	10624163
1050	10762500	10785578	10808663	10831758	10854850

y'	5	6	7	8	9
1000	00100075	00120108	00140147	00160192	00180243
1001	00300675	00320768	00340867	00360973	00381084
1002	00501876	00522030	00542189	00562354	00582525
1003	00703679	00723893	00744112	00764337	00784569
1004	00906084	00926358	00946637	00966923	00987215
1005	01109092	01129426	01149766	01170112	01190464
1006	01312702	01333907	01353497	01373903	01394316
1007	01516917	01537372	01557833	01578299	01598772
1008	01721736	01742252	01762775	01783300	01803833
1009	01927160	01947736	01968318	01988906	02009500
1010	02133191	02153827	02174470	02195118	02215773
1011	02339827	02360524	02381227	02401936	02422652
1012	02547070	02567828	02588592	02609317	02630138
1013	02754921	02775740	02796564	02817395	02838232
1014	02963380	02984259	03005145	03026036	03046934
1015	03172447	03193388	03214334	03235286	03256245
1016	03382124	03403152	03424133	03445146	03466166
1017	03592411	03613473	03634542	03655616	03676697
1018	03803308	03824431	03845561	03866965	03887838
1019	04014816	04036001	04057192	04078388	04099591
1020	04226937	04248182	04269434	04290692	04311956
1021	04439669	04460976	04482289	04503608	04524933
1022	04653014	04674382	04695757	04717137	04738524
1023	04866973	04888402	04909838	04931280	04952728
1024	05081546	05103036	05124534	05146037	05167547
1025	05296733	05318286	05339844	05361409	05382980
1026	05512534	05534150	05555770	05577397	05599030
1027	05728955	05750630	05772312	05794000	05815695
1028	05945990	05967727	05989471	06011221	06032977
1029	06163642	06185441	06207247	06229058	06250876
1030	06381912	06403773	06425640	06447514	06469393
1031	06600800	06622723	06644653	06666588	06688529
1032	06820308	06842293	06864284	06886281	06908284
1033	07040435	07062481	07084534	07106593	07128659
1034	07261181	07283290	07305404	07327526	07349654
1035	07482549	07504720	07526899	07549080	07571267
1036	07704538	07726771	07749010	07771256	07793597
1037	07927148	07949114	07971745	07994053	08016367
1038	08150382	08172739	08195103	08217473	08239849
1039	08374238	08396658	08419084	08441516	08463955
1040	08598718	08621200	08643489	08666184	08688685
1041	08823822	08846367	08868918	08891475	08914039
1042	09049552	09072159	09094772	09117392	09140018
1043	09275906	09298576	09321252	09343935	09366623
1044	09502887	09525627	09548358	09571103	09593855
1045	09730495	09753290	09776091	09798900	09821713
1046	09958729	09981587	10004452	10027322	10050199
1047	10187592	10210513	10233440	10256374	10279315
1048	10417083	10440067	10463057	10486053	10509056
1049	10647204	10670250	10693303	10716363	10737428
1050	10877954	10901063	10924179	10947301	10970430

γ′	0	1	2	3	4
1051	10993565	11016706	11039854	11063008	11086168
1052	11225261	11248465	11271676	11294896	11318116
1053	11457588	11480855	11504129	11527409	11550695
1054	11690546	11713877	11737214	11760557	11783907
1055	11924138	11947531	11970932	11994338	12017751
1056	12158362	12181819	12205282	12228752	12252229
1057	12393219	12416740	12440267	12463800	12487340
1058	12628711	12652295	12775886	12699483	12723086
1059	12864838	12888486	12912139	12935800	12959466
1060	13101600	13125311	13149029	13172753	13196483
1061	13338998	13362773	13386554	13410342	13434136
1062	13577033	13600871	13624716	13648567	13672425
1063	13815705	13839607	13863516	13887431	13911352
1064	14055014	14078980	14109953	14126932	14150917
1065	14294963	14318992	14343029	14367072	14391121
1066	14535550	44559643	14583744	14607850	14631963
1067	14776776	14800934	14825098	14849269	14873446
1068	15018643	15042865	15067093	15091328	15115569
1069	15261151	15285437	15309729	15334028	15358334
1070	15504300	15528650	15553007	15577370	15601739
1071	15748091	15772506	15796926	15821354	15845787
1072	75992525	16017004	16041489	16065980	16090478
1073	16237602	16262145	16286694	16311250	16335813
1074	16483322	16507930	16532544	16557164	16581791
1075	16729688	16754359	16779038	16803723	16828414
1076	16976698	17001434	17026177	17050926	17075682
1077	17224353	17249154	17273962	17298776	17323596
1078	17472655	17497521	17522393	17547272	17572157
1079	17721604	17746534	17771471	17796415	17821365
1080	17971200	17996195	18021197	18046201	18071220
1081	18221444	18256504	18271571	18296644	18321723
1082	18472337	18497462	18522593	18547731	18572876
1083	18723879	18749069	18774265	18799468	18824677
1084	18976070	19001325	19026587	19051855	19077129
1085	19228913	19254233	19279559	19304892	19330232
1086	19482406	19507791	19533182	19558581	19583986
1087	19736550	19762001	19787457	19812921	19838391
1088	19979135	20016863	20042345	20067914	20093449
1089	20246797	20272378	20297965	20323559	20349160
1090	20502900	20528546	20554200	20579858	20605524
1091	20759657	20785369	20811087	20836812	20862548
1092	21017069	21042846	21068630	21094420	21120217
1093	21275136	21300978	21326828	21352684	21378546
1094	21533858	21559767	21585682	21611603	21637581
1095	21793238	21819212	21845192	21871179	21897173
1096	22053274	22079313	22105360	22131413	22157472
1097	22313967	22340073	22366185	22392304	22418429
1098	22575319	22601491	22627669	22653853	22680044
1099	22837330	22863567	22889811	22916062	22942319
1100	23100000	23126303	23152613	23178930	23205253
1101	23363330	23389700	23416075	23442458	23468847
1102	23627321	23653756	23680198	23706647	23733102

y'	5	6	7	8	9
1051	11109334	11132507	11155686	11178871	11202063
1052	11341545	11364581	11387823	11411072	11434327
1053	11573988	11597287	11620592	11643904	11667222
1054	11807263	11830625	11853994	11877369	11900750
1055	12041170	12064596	12088028	12111466	12134911
1056	12275711	12299200	12322695	12346197	12369705
1057	12510886	12534438	12557997	12581562	12605133
1058	12746695	12770311	13793933	12817562	12841197
1059	12983139	13006819	13030505	13054197	13077895
1060	13220220	13243963	13267712	13291468	13315230
1061	13457936	13481742	13505556	13529375	13553201
1062	13696289	13720159	13744036	13767919	13791809
1063	13935280	15959214	13983154	14007101	14031055
1064	14174909	14198907	14222911	14236922	14270939
1065	14415176	14439238	14463306	14487381	14511462
1066	14656083	14680209	14704341	14728480	14752625
1067	14897630	14921820	14946016	14970219	14994428
1068	15139817	15164071	15188331	15212598	15236871
1069	15382645	15406963	15431288	15455619	15479956
1070	15626115	15650498	15674886	15699285	15723683
1071	15870228	15894674	15919127	15943587	15918052
1072	16114983	16139494	16164011	16188535	16213065
1073	16360382	16384957	16409539	16434127	16458721
1074	16606424	16631064	16655710	16680363	16705022
1075	16853112	16877816	16902527	16927244	16951968
1076	17100445	17125214	17149989	17174770	17199559
1077	17348423	17373257	17398097	17422943	17447796
1078	17597049	17621947	17646851	17671762	17696680
1079	17846321	17871284	17896263	17921229	17946211
1080	18096241	18121269	18146303	18171343	18196391
1081	18346809	18371902	18397001	18422106	18447218
1082	18598027	18623184	18648348	18673518	18698615
1083	18849893	18875116	18900345	18925580	18950822
1084	19102410	19127698	19152992	19178292	19203560
1085	19355578	19380930	19406289	19431655	19457027
1086	19609396	19664814	19660238	19685669	19711106
1087	19863867	19889350	19914840	19940336	19965838
1088	20118990	20144539	20170093	20195655	20221223
1089	20374767	20400380	20426000	20451627	20477260
1090	20631197	20656876	20682561	20708253	20733952
1091	20888281	20914026	20939777	20965534	20991298
1092	21146020	21171830	21197647	21223470	21249300
1093	21404415	21430291	21456173	21482061	21507957
1094	21663466	21689407	21715355	21741309	21767270
1095	21923173	21949180	21975194	22001214	22027240
1096	22183538	22209611	22265690	22261796	22287868
1097	22444561	22470699	22496844	23522996	22549154
1098	22706242	22732447	22758657	22784875	22811099
1099	22968582	22994853	23021130	23047418	23073703
1100	23231583	23257919	23284262	23310611	23336967
1101	23495243	23521645	23548054	23574470	23600892
1102	23759564	23786033	23812508	23838989	23865478

y′	5	6	7	8	9
1103	27901973	23918474	23944982	23971497	23998019
1104	24157286	24183854	24210429	24237010	24263597
1105	24423263	24449897	24476537	24503185	24529839
1106	24689902	24716602	24743309	24770023	24796743
1107	24957204	24983971	25010745	25037525	25064311
1108	25225171	25252004	25278844	25305691	25332544
1109	25493803	25520703	25547609	25574522	25601442
1110	25763100	25790066	25817039	25844019	25871001
1111*	26033063	26060096	26087136	26114182	26141235
1112	26303693	26330792	26357899	26385012	26412131
1113	26574990	26602156	26629329	26656509	26683695
1114	26846954	26874188	26901428	26928674	26955927
1115	27119588	27146888	27174194	27201508	27228828
1116	27392890	27420257	27447630	27475011	27502398
1117	27666861	27694295	27721736	27749185	27776638
1118	27941503	27969004	27996512	28024027	28051548
1119	28216816	28244384	28271959	28299541	28327129
1120	28492800	28520435	28548077	28575726	28603382
1121	28769456	28797159	28824868	28852584	28880307
1122	29046784	29074555	29102331	29130115	29157905
1123	29324787	29352624	29380468	29408319	29436176
1124	29603462	29631367	29659278	29687197	29715121
1125	29882813	29910785	29938764	29966749	29994742
1126	30162838	30190877	30218924	30246977	30275037
1127	30443538	30471646	30499760	30527880	30556008
1128	30724915	30753090	30781272	30809460	30837655
1129	31006969	31035312	31063461	31091717	31119980
1130	31289700	31318010	31346328	31374652	31402982
1131	31573109	31701487	31629872	31658264	31686663
1132	31857197	31885643	31914096	31942556	31971022
1133	32141964	32170478	32198999	32227526	32256061
1134	32427410	32455992	32484581	32513177	32541780
1135	32713538	32742188	32770845	32799508	32828179
1136	33000346	33029064	33057789	33086521	33115260
1137	33287835	33316622	33345415	33374215	33403022
1138	33576007	33604862	33633723	33662592	33691467
1139	33864862	33893785	33922715	33951652	33980595
1140	34154400	34183391	34212390	34241395	34270407
1141	34444622	34473682	34502749	34531822	34560903
1142	34735529	34764657	34793792	34822934	34852083
1143	35027121	35056318	35085521	35114732	35143949
1144	35319398	35348664	35377936	35407216	35436502
1145	35612363	35641697	35671038	35700386	35729740
1146	35906014	35935417	35964826	35994243	36023667
1147	36200352	36229824	36269303	36288788	36318280
1148	36495379	36524920	36554467	36584022	36613583
1149	36791095	36820704	36850321	36879944	36909574
1150	37087500	37117178	37146864	37176556	37206255
1151	37384595	37414343	37444097	37473858	37503626
1152	37682380	37712197	37742021	37771851	37801689
1153	37980858	38010743	38040636	38070636	38100442
1154	38280026	38309981	38339943	38369912	38399888

y'	5	6	7	8	9
1103	24024547	24051081	24077623	24104171	24130725
1104	24290192	24316792	24343400	24370014	24396635
1105	24556499	24583166	24609840	24636521	24663208
1106	24823470	24850204	24876944	24903691	24930444
1107	25091105	25117905	25144711	25171525	25198045
1108	25359404	25386270	25413144	25440023	25466910
1109	25628368	25655301	25682241	25709187	25736140
1110	25897998	25924998	25952004	25979017	26006037
1111	26168295	26195361	26222434	26249514	26276600
1112	26439258	26466391	26493531	26520677	26547830
1113	26710889	26738088	26765295	26792508	26819728
1114	26983187	27010454	27037727	27065007	27092294
1115	27256155	27283488	27310829	27338176	27365529
1116	27529792	27557192	27584599	27612013	27639434
1117	27804098	27831566	27859040	27886521	27914009
1118	28079076	28106610	28134152	28161700	28189254
1119	28354724	28382326	28409934	28437549	28465171
1120	28631044	28658713	28686389	28714071	28741760
1121	28908036	28935773	28963516	28991265	29019022
1122	29185702	29213505	29241315	29269132	29296956
1123	29464040	29491911	29519789	29547673	29575564
1124	29743053	29770991	29798937	29826888	29854847
1125	20022741	30050747	30078759	20106779	30134805
1126	30303103	30331177	30359257	30387344	30415438
1127	30584142	30612283	30640431	30668586	30696747
1128	30865857	30894066	30922282	30950504	30978733
1129	31148250	31176526	31204810	31233100	31261396
1130	31431320	31459664	31488015	31516373	31544738
1131	31715068	31743480	31771899	31800325	31828757
1132	31999495	32027975	32056462	32084953	32113456
1133	32284602	32313150	32341705	32370267	32398835
1134	32570389	32599005	32627628	32656258	32684894
1135	32856856	32885541	32914232	32942929	32971634
1136	33144005	33172758	33201517	33230283	33259055
1137	33431836	33460657	33489484	33518318	33547159
1138	33720349	33749238	33778154	33807036	33835946
1139	34009545	34038503	34067467	34096438	34125415
1140	34299426	34328451	34357484	34386523	34415569
1141	34589990	34619084	34648185	34677293	34706407
1142	34881259	34910402	34939571	34968747	34997931
1143	35173174	35202405	35231643	35260888	35290140
1144	35465795	35495094	35524401	35553715	35583035
1145	35759102	35788471	35817846	35847228	35876618
1146	36053097	36082534	36111978	36141430	36170887
1147	36347780	36377286	36406799	36436319	36465846
1148	26643151	36672726	36702308	36731897	36761492
1149	36939211	36968855	36998506	37028164	37057828
1150	37235961	37265674	37295394	37325121	37354855
1151	37533402	37563184	37592978	37622768	37652571
1152	37831533	37861384	37891242	37921107	37950979
1153	38130356	38160276	38190206	38220137	38250078
1154	38429870	38450960	38489856	38519860	38549870

ARTICLE XII.

New Properties of the Conic Sections.

By Mr. J. F. W. Herschel, *St. John's College, Cambridge.*

To the Editor of the Mathematical Repository.

Sir,

As the following properties of the conic sections appear to me, so far as my reading has hitherto extended, to have been unnoticed, I have been induced to trouble you with them, under the idea that you may, perhaps, not deem them unworthy a place in your Repository.

If we call a the semi-axis major of a conic section, ae the excentricity, r, the distance between one focus and a point in the curve, R, the distance between the other, and the same point, and $u =$ the angle included between r and the line joining the foci, we have

$$r = a\,\frac{1-e^2}{1-e\cos u}, \text{ and } \therefore\ R = 2a - r = a\,\frac{1-2e\cos u+e^2}{1-e\cos u};$$

$$\therefore\ \frac{R}{r} = \frac{1-2e\cos u+e^2}{1-e\cos u} \times \frac{1-e\cos u}{1-e^2} = \frac{1-2e\cos u+e^2}{1-e^2}.$$

For u substitute successively $\theta,\ \theta+\frac{2\pi}{n},\ \theta+\frac{4\pi}{n} \ldots\ \theta+\frac{(n-1)2\pi}{n}$;

and call the corresponding values of r, R, $\left\{\begin{matrix} r(1), r(2), \ldots r(n), \text{ and} \\ R(1), R(2), \ldots R(n) \end{matrix}\right\}$

and the product of the values $\frac{R}{r}$ will be $\frac{R(1)\,.\,R(2)\,\ldots\,R(n)}{r(1)\,.\,r(2)\,\ldots\,r(n)} =$

$$\frac{(1-2e\cos\theta+e^2)\,(1-2e\cos(\theta+\frac{2\pi}{n})+e^2)\ldots(1-2e\cos(\theta+\frac{(n-1).2\pi}{n})+e^2)}{(1-e^2)^n}$$

$$= \frac{1-2e^n.\cos n\theta+e^{2n}}{(1-e^2)^n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

which is the general equation I proposed to deduce.

But when one of the lines, which drawn from the focus, divides the circumference into n parts, (i. e. of the lines $r(1), r(2)$, &c.) has its other extremity in the farther vertex, assuming this for $r(1)$; we have $\theta = 0$, and $\therefore \cos n\theta = 1$, and the equation (A) becomes $\frac{R(1)\,.\,R(2)\ldots R(n)}{r(1)\,.\,r(2)\ldots\,r(n)} = \frac{1-2e^n+e^{2n}}{(1-e^2)^n} = \frac{(1-e^n)^2}{(1-e^2)^n}$, which geometrically expressed, would afford a property not inelegant of lines drawn from the focus of any conic section to its periphery, dividing it into n parts, and making equal angles at the focus, one of which is the greatest focal distance. It may be remarked in general, that in the hyperbola, since a line drawn to the focus as (r) when it makes with the line joining the foci an angle whose cosine is $-\frac{1}{e^2}$, is parallel to the asymptote, the intersections of $r(1)$, $r(2)$, &c. with the curve will lie in the *same* hyperbola, so long as arc, $\cos -\frac{1}{e^2}$ is greater, or $2\pi -$ arc, $\cos -\frac{1}{e^2}$ less, than $\theta + \frac{(n-m)2\pi}{n}$, but when m is such that $\theta + \frac{(n-m)2\pi}{n}$ is between arc, $\cos -\frac{1}{e^2}$ and $2\pi -$ arc, $\cos -\frac{1}{e^2}$; the intersections, or the extremities of the r will pass into the opposite hyperbola.

Now in the equation (A) let n be odd, and let one point of division of the conic section lie in the *nearer* vertex to the focus, the origin of the r; we have $\theta = 180° = \pi$, and

$$\frac{\text{R}(1) \cdot \text{R}(2) \cdot \text{R}(3) \ldots\ldots \text{R}(n)}{r(1) \cdot r(2) \cdot r(3) \ldots\ldots r(n)} = \frac{1 + 2e^n + e^{2n}}{(1-e^2)^n} = \frac{(1+e^n)^2}{(1-e^2)^n},$$

which expresses another general property of lines drawn, from one focus, multisecting the periphery, so that the angles at the focus are equal, one of the divisions falling on the nearer vertex.

These are the theorems, Sir, which it was my design to demonstrate. It would at first view appear that they are analogous to Cotes's properties of the circle, but a slight consideration shews them to be totally separated from those forms. It is almost too obvious to require that I should mention that the same equations, *with all the generality of the algebraic signs*, apply to the ellipse, hyperbola and parabola. In the former case, a is positive and $e < 1$. In the second a is negative and $e > 1$. In the third, $a = \infty$ and $e = 1$. If we would however, geometrically neglect the proper signs, we shall have

in the hyperbola $\frac{\text{R}(1) \cdot \text{R}(2) \ldots \text{R}(n)}{r(1) \cdot r(2) \ldots r(n)} = \frac{1 - 2e^n \cos\theta + e^{2n}}{(e^2-1)^n}$;

and in the other two cases,

$$\frac{\text{R}(1) \cdot \text{R}(2) \ldots\ldots \text{R}(n)}{r(1) \cdot r(2) \ldots\ldots r(n)} = \frac{(e^n-1)^2}{(e^2-1)^n} \text{ and } \frac{(e^n+1)^2}{(e^2-1)^n}.$$

ARTICLE XIII.

INDETERMINATE PROBLEMS. *By Mr.* CUNLIFFE.

PROBLEM 1. To find the sides of three right-angled triangles in rational numbers, whose areas shall be equal to each other.

SOLUTION. It is known that $m^2 - n^2$, $2mn$ and $m^2 + n^2$, express the sides of a right-angled triangle: and suppose $p^2 - q^2$, $2pq$ and $p^2 + q^2$, denote the sides of another right-angled triangle of equal area; then will $mn\,(m^2 - n^2) = pq\,(p^2 - q^2)$.

Put $p = m + r$ and $q = n - r$, then $pq = mn - r(m-n) - r^2$, and $p^2 - q^2 = m^2 - n^2 + 2r(m+n)$; therefore $mn\,(m^2 - n^2) = pq(p^2 - q^2)$ $= (mn - r(m-n) - r^2)(m^2 - n^2 + 2r(m+n)) = mn(m^2 - n^2) - r(m+n)$ $\times (m^2 - 4mn + n^2) - 3r^2\,(m^2 - n^2) - 2r^3(m+n)$; which becomes $2r^2 + 3r(m-n) = -m^2 + 4mn - n^2$; completing the square, &c. $r = \pm \frac{1}{4}\sqrt{(m^2 + 14mn + n^2)} - \dfrac{3(m-n)}{4}$. From whence it appears that, in order to have r rational, $m^2 + 14mn + n^2$ must be a rational square. Put $m^2 + 14mn + n^2 = (m + n + s)^2 = m^2 + 2mn + n^2 + 2s(m+n) + s^2$, whence $m = \dfrac{2sn + s^2}{12n - 2s}$; and supposing the positive sign to have place $r = \frac{1}{4}\sqrt{(m^2 + 14mn + n^2)} - \dfrac{3(m-n)}{4} = \dfrac{m + n + s - 3(m-n)}{4} = \dfrac{4n + s - 2m}{4} = \dfrac{12n^2 - s^2}{12n - 2s}$, or if the negative sign in the value of r, is supposed to take place, $r = -\frac{1}{4}\sqrt{(m^2 + 14mn + n^2)} - \dfrac{3(m-n)}{4} = \dfrac{2n - s - 4m}{4}$ $= \dfrac{12n^2 - 12ns - s^2}{2 \times (12n - 2s)}$.

Now as r has two values, it is evident from what has been done, that we can find the sides of three right-angled triangles in rational numbers, whose areas are equal to each other.

Ex. Take $n = 1$ and $s = 3$ then $m = \dfrac{5}{2}$, $2mn = 5$, $m^2 - n^2 = \dfrac{21}{4}$ which are the legs of one triangle, and taking the first expression for r, viz. $r = \dfrac{12n^2 - s^2}{12n - 2s} = \frac{1}{2}$; then $p = m + r = 3$, $q = n - r = \frac{1}{2}$, $2pq = 3$, $p^2 - q^2 = \dfrac{35}{4}$, which are the legs of another right-angled triangle whose area is equal to the former.

Taking the second value of r, viz. $r = \frac{12n^2 - 12ns - s^2}{2 \times (12n - 2s)} = -\frac{33}{12} = -\frac{11}{4}$, $p = m + r = -\frac{1}{4}$, $q = n - r = \frac{15}{4}$; $2pq = -\frac{15}{8}$ and $p^2 - q^2 = -\frac{224}{16} = -14$ or $\frac{15}{8}$ and 14 are the legs of a third right-angled triangle whose area is equal to that of the former: And reducing the numbers to a common denominator, we shall have the legs of one triangle 40, 42; of another 24, 70: and of a third 15, 112; all of which have the same area. And from the preceding general solution, innumerable sets of right-angled triangles may be found, three in each set, that will have the same area.

Other general values for p, q; m, n may be obtained as follows. Put $m^2 + 14mn + n^2 = \left(\frac{am}{b} - n\right)^2 = \frac{a^2m^2}{b^2} - \frac{2amn}{b} + n^2$, whence $n = \frac{m \times (a^2 - b^2)}{14b^2 + 2ab}$; take $m = 14b^2 + 2ab$; then $n = a^2 - b^2$; whence $r = \pm\frac{1}{4}\sqrt{(m^2 + 14mn + n^2)} - \frac{3(m-n)}{4} = \pm\frac{1}{4}\left(\frac{am}{b} - n\right) - \frac{3(m-n)}{4}$ and, supposing the affirmative sign to have place, $r = \frac{m(a-3b) + 2bn}{4b} = a^2 + 2ab - 11b^2$, $p = (m + r) = a^2 + 4ab + 3b^2$, $q = n - r = 10b^2 - 2ab$. But supposing the negative sign to take place, $r = \frac{4bn - m(a+3b)}{4b} = \frac{a^2 - 10ab - 23b^2}{2}$, $p = m + r = \frac{a^2 - 6ab + 5b^2}{2}$, $q = n - r = \frac{a^2 + 10ab + 21b^2}{2}$.

Ex. Take $a = 2$, $b = 1$, then $m = 18$, $n = 3$; $p = 15$ and $q = 6$ or $p = -\frac{3}{2}$ and $q = \frac{45}{2}$; whence the legs of the three right-angled triangles are 108, 315; 180, 189; $\frac{135}{2}$ and 504.

Another way of finding the sides of three right-angled triangles, in rational numbers, having the same area, may be as follows:

Let $m^2 - n^2$, $2mn$ and $m^2 + n^2$, denote the sides of one of the triangles; and $p^2 - q^2$, $2pq$ and $p^2 + q^2$ the sides of another: then as the areas of these triangles are to be equal to each other, we shall have $mn(m^2 - n^2) = pq(p^2 - q^2)$.

Put $m=p$, and the equation becomes $mn\,(m^2-n^2)=mq(m^2-q^2)$, which gives $m^2=\frac{n^3-q^3}{n-q}=n^2+nq+q^2$, whence $n^2+nq+\frac{q^2}{4}=m^2-\frac{3q^2}{4}=\frac{4m^2-3q^2}{4}$; taking the square roots, &c. $n=\pm\frac{1}{2}\sqrt{(4m^2-3q^2)}-\frac{q}{2}$; whence it appears that, in order to have n rational, $4m^2-3q^2$ must be a rational square. Put $4m^2-3q^2=\left(\frac{aq}{b}-2m\right)^2=\frac{a^2q^2}{b^2}-\frac{4aqm}{b}+4m^2$, which gives $m=\frac{q(a^2+3b^2)}{4ab}$. Put $q=4ab$, then $m=a^2+3b^2$; and supposing the affirmative sign to take place, in the value of n, $n=\frac{1}{2}q\times\frac{(a-b)}{b}-m=a^2-2ab-3b^2$.

Again, supposing the negative sign to take place in the value of n, $n=m-\frac{1}{2}q\times\frac{(a+b)}{b}=3b^2-2ab-a^2$, in which expressions a and b may be taken at pleasure.

Ex. Take $a=2$ and $b=1$, then $m=p=7$, and $q=8$, $2pq=112$, $p^2-q^2=-15$; and the value of n by the first expression is -3, whence $2mn=-42$, $m^2-n^2=40$.

Again, the value of n from the second expression is -5, therefore $2mn=-70$, $m^2-n^2=24$.

One of the legs in each of the three triangles in the preceding example happens to be negative, they may, therefore, without detriment, be all taken affirmative; that is the sides of one of the triangles will be 15, 112 and 113; the sides of another 40, 42 and 58; and the sides of the third 24, 70 and 74.

Ex. 2. Take $a=1$ and $b=2$, then $m=p=13$, $q=8$, $2pq=208$, $p^2-q^2=105$.

The first expression for n, gives $n=-15$, whence $2mn=-390$, and $m^2-n^2=-56$; and the second expression for n gives $n=7$; then $2mn=182$, $m^2-n^2=120$. And the sides of these three triangles are, 1st. triangle 105, 208 and 233; 2d. 56, 390 and 394; 3d. 120, 182 and 218.

Other general expressions for m, n; p, q, will be obtained as follows:

Put $4m^2-3q^2=(2m-d)^2=4m^2-4md+d^2$, whence $m=\frac{3q^2+d^2}{4d}$, $2m-d=\frac{3q^2-d^2}{2d}$; and hence, $n=\pm\frac{1}{2}(2m-d)-\frac{q}{2}$;

therefore $n = \frac{3q^2-2dq-d^2}{4d}$, or $n = \frac{d^2-2dq-3q^2}{4d}$, where q and d may be taken at pleasure.

Ex. Take $q=3$, and $d=1$, then $m=p=7$, and $n=5$, or $n=-8$, $2pq=42$, $p^2-q^2=40$, $p^2+q^2=58$; taking $n=5$. $2mn=70$, $m^2-n^2=24$ and $m^2+n^2=74$, or taking $n=-8$, $2mn=-112$, $m^2-n^2=-15$ and $m^2+n^2=113$, and because both legs are negative, in the latter circumstance, we may put $2mn=112$, $m^2-n^2=15$ and $m^2+n^2=113$.

PROBLEM II. To find the sides of any number of right-angled triangles, in rational numbers, whose areas shall be equal to each other.

SOLUTION. Let a, b and h, denote the two legs, and the hypothenuse of a given right-angled triangle in rational numbers: also let a', b' and h', denote the two legs and hypothenuse, in rational numbers, of another right-angled triangle of equal area with the given triangle, that is $a'b'=ab$, or $a'=\frac{ab}{b'}$; put $b'=b+s$; then $a'=\frac{ab}{b+s}$ and hence $h'^2=a'^2+b'^2$ $=\frac{a^2b^2}{(b+s)^2}+(b+s)^2=\frac{a^2b^2+(b+s)^4}{(b+s)^2}=$ a square, therefore $a^2b^2+(b+s)^4=a^2b^2+b^4+4b^3s+6b^2s^2+4bs^3+s^4=b^2h^2+4b^3s+6b^2s^2+4bs^3+s^4=$ a square $=(bh+\frac{2b^2s}{h}-s^2)^2=b^2h^2+4b^3s+\frac{4b^4s^2}{h^2}-2bhs^2-\frac{4b^2s^3}{h}+s^4$; whence $s=\frac{2b^3-3bh^2-h^3}{2h(h+b)}$, $b'=b+s=\frac{2b^2-h^2}{2h}$, $a'=\frac{ab}{b+s}=\frac{2abh}{2b^2-h^2}$, $h'^2=a'^2+b'^2$ $=\frac{(h^4+4b^2h^2-4b^4)^2}{4h^2(2b^2-h^2)^2}$, or $h'=\frac{h^4+4b^2h^2-4b^4}{2h(2b^2-h^2)}$; and consequently $a''=\frac{2a'b'h'}{2b'^2-h'^2}$, $b''=\frac{2b'^2-h'^2}{2h'}$ and $h''=\frac{h'^4+4b'^2h'^2-4b'^4}{2h'(2b'^2-h'^2)}$, &c.

The way of obtaining the sides of the triangles, in succession, is abundantly evident from what has been done.

The most simple numerical values for the sides of a right-angled triangle, in rational numbers, are 3, 4 and 5, therefore we may take $b=4$ and $h=5$, whence $a'=\frac{120}{7}$, $b'=\frac{7}{10}$ and $h'=\frac{1201}{70}$.

But the sides of a third right-angled triangle derived in succession, in the same manner, from the last, will evidently be large fractional numbers.

In order, therefore, to obtain numbers moderately small for the sides of six right-angled triangles of equal area, I have employed one of the sets of numbers for three right-angled triangles, deduced in the examples annexed to the first problem, and from each of the three triangles deduced another by means of the conclusions in the present problem.

Ex. 40, 42, and 58, are numbers expressing the sides of a right-angled triangle, therefore we may put $a = 40$, $b = 42$ and $h = 58$. And by means of the preceding conclusions, we shall have $\frac{41}{29}$, $\frac{48720}{41}$ and $\frac{1412881}{1189}$ for the sides of another right-angled triangle of equal area.

Secondly, 24, 70 and 74 are numbers expressing the sides of a right-angled triangle, equal in area to that of the former, therefore we may put, $a = 24$, $b = 70$ and $h = 74$: and from these we shall have $\frac{1081}{37}$, $\frac{62160}{1081}$ and $\frac{2579761}{39997}$ for the sides of another right-angled triangle of equal area.

Thirdly, 15, 112 and 113 are the sides of a third right-angled triangle, equal in area to each of the two former, therefore we may put, $a = 15$, $b = 112$ and $h = 113$; and from these we shall get $\frac{12319}{226}$, $\frac{379680}{12319}$ and $\frac{174336961}{2784094}$ for the sides of another right-angled triangle of equal area.

Another expression for s, and consequently for the side of the triangle $b + s$, may be found in the following manner.

The general expression to be made a square, is $b^2h^2 + 4b^3s + 6b^2s^2 + 4bs^3 + s^4$; therefore put $b^2h^2 + 4b^3s + 6b^2s^2 + 4bs^3 + s^4 = (bh + \frac{2b^2s}{h} + gs^2)^2 = b^2h^2 + 4b^3s + \frac{4b^4s^2}{h^2} + 2gbhs^2 + \frac{4gb^2s^3}{h} + g^2s^4$; put $2gbh + \frac{4b^4}{h^2} = 6b^2$, or $g = \frac{3bh^2 - 2b^3}{h^3}$, in order to take away the three leading terms on each side: then from the equality of the remaining terms, viz. $4bs^3 + s^4 = \frac{4gb^2s^3}{h} + g^2s^4$, $s = 4b\frac{(h - gb)}{h(g^2 - 1)}$, $b + s = \frac{hb(g^2 + 3) - 4gb^2}{h(g^2 - 1)} = \frac{b(4b^4 - 3h^4)}{4b^4 - 8b^2h^2 + h^4} = b'$, and $a' = \frac{ab}{b'} = \frac{a(4b^4 - 8b^2h^2 + h^4)}{4b^4 - 3h^4}$.

Problem III. To find any number of rational squares h^2, h'^2, h''^2, &c. ad libitum, together with a certain number N, which being either added to, or subtracted from, each of the said squares, the sums and differences thence arising shall all be rational squares.

Solution. Let a, b and h; a', b' and h'; a'', b'' and h'', &c. denote the two legs and hypothenuse, in rational numbers, of the same number of right-angled triangles, of equal area, as is the required number of squares; then by the nature of the question, $ab = a'b' = a''b''$, &c. or $2ab = 2a'b' = 2a''b''$, &c. and by the known property of a right-angled triangle $h^2 = a^2 + b^2$, $h'^2 = a'^2 + b'^2$; $h'^2 = a''^2 + b''^2$, &c.

Put $N = 2ab = 2a'b' = 2a''b''$, &c. = four times the common area, then $h^2 \pm N = a^2 \pm 2ab + b^2 = (a \pm b)^2$; $h'^2 \pm N = a'^2 \pm 2a\,b' + b'^2 = (a' \pm b')^2$; $h''^2 \pm N = a''^2 \pm 2a''b'' + b''^2 = (a'' \pm b'')^2$, &c. are evidently all squares.

Ex. Let it be required to find three rational squares, and a certain number N, which being either added to or subtracted from any one of the said squares, the sum or difference shall be a square.

It was found, in one of the examples annexed to the first problem, that the following numbers express the sides of three right-angled triangles of equal area, viz.

40, 42 and 58; 24, 70 and 74; 15, 112 and 113;
whence $N = 3360$, and therefore $58^2 \pm 3360$; $74^2 \pm 3360$ and $113^2 \pm 3360$ are all squares.

It moreover appears from the conclusions of the present problem, and the second example to problem 2, that the following expressions are all rational squares, viz.

$$58^2 \pm 3360;\ 74^2 \pm 3360;\ 113^2 \pm 3360;\ \left(\frac{1412881}{1189}\right)^2 \pm 3360;$$

$$\left(\frac{2579761}{39997}\right)^2 \pm 3360;\ \text{and}\ \left(\frac{174336961}{2784094}\right)^2 \pm 3360.$$

The way of extending the problem to any assigned number of squares, is sufficiently evident from what has been done.

ARTICLE XIV.

A method of finding two Arcs of the same Ellipse or Hyperbola whose sum, or difference, shall be a finite Algebraical Expression. *By Mr.* CUNLIFFE.

First for the Ellipse.

Let e denote the excentricity of an ellipse whose semi-transverse axis is 1; and let x denote a part of the transverse axis intercepted between the centre and a perpendicular ordinate: also let E denote the arc of the ellipse intercepted between the conjugate axis and the said ordinate. Then it is well known that $\dot{E} = \dot{x}\sqrt{\frac{1 - e^2x^2}{1 - x^2}}$. Put $ex' = \sqrt{\frac{1 - e^2x^2}{1 - x^2}}$; whence $\dot{x} = \frac{1}{e}\sqrt{\frac{1 - e^2x'^2}{1 - x'^2}}$; then will $\dot{E} = ex'\dot{x}$; taking the *fluents*, $E = exx' - \int ex\dot{x}' = exx' - \int \dot{x}'\sqrt{\frac{1 - e^2x'^2}{1 - x'^2}} = exx' - E'$, or $E + E' = exx' = x\sqrt{\frac{1 - e^2x^2}{1 - x^2}}$! Where E′ denotes an arc of the the same ellipſe, intercepted between the conjugate axis and a parallel ordinate at the distance $x' = \frac{1}{e}\sqrt{\frac{1 - e^2x^2}{1 - x^2}}$ from the centre.

The last expression is very remarkable, as it furnishes the means of determining two arcs of the same ellipse whose sum is expressible by an algebraical quantity.

Again, putting $ex'' = \sqrt{\frac{1 - e^2x'^2}{1 - x'^2}}$, &c. &c. and proceding as before we shall pretty obviously have $E' = ex'x'' - E''$; $E'' = ex''x''' - E'''$; $E''' = ex'''x'''' - E''''$, &c. where, E'', E''', E'''', &c. denote arcs of the same ellipse intercepted between the conjugate axis, and parallel ordinates at the respective distances of $x'' = \frac{1}{e}\sqrt{\frac{1 - e^2x'^2}{1 - x^2}}$, $x''' = \frac{1}{e}\sqrt{\frac{1 - e^2x''^2}{1 - x''^2}}$, $x'''' = \sqrt{\frac{1 - e^2x'''^2}{1 - x'''^2}}$, &c. and consequently $E = exx' - ex'x'' + ex''x''' - ex'''x''''$, &c. to n terms $\pm E^{(n)}$, or $E \mp E^{(n)} = e \times (xx' - x'x'' + x''x''' - x'''x''''$, &c. to n terms), where $E^{(n)}$ denotes an arc of the same ellipse intercepted between the conjugate axis and parallel ordinate at the distance $x^{(n)}$ from the centre; the upper or lower sign takes place accordingly as n is even or odd.

Secondly for the Hyperbola.

Let e denote the excentricity of an hyperbola, whose semi-transverse axis is 1; and let x denote a part of the transverse axis intercepted between the centre, and a perpendicular ordinate; also let H denote the arc of the hyperbola intercepted between the principal vertex and the said ordinate: then it is well known that $\dot{H} = \dot{x}\sqrt{\frac{e^2x^2-1}{x^2-1}}$. Put $ex' = \sqrt{\frac{e^2x^2-1}{x^2-1}}$, whence $x = \frac{1}{e}\sqrt{\frac{e^2x'^2-1}{x'^2-1}}$, then will $\dot{H} = \dot{x}\sqrt{\frac{e^2x^2-1}{x^2-1}} = ex'\dot{x}$; taking the fluents, $H = e\dot{x}x' - \int ex\dot{x}' = exx' - \int \dot{x}'\sqrt{\frac{e^2x'^2-1}{x'^2-1}} = exx' - H'$ or $H + H' = exx' = x\sqrt{\frac{e^2x^2-1}{x^2-1}}$; where H′ denotes an arc of the same hyperbola, intercepted between the principal vertex and a perpendicular ordinate at the distance of $x' = \frac{1}{e}\sqrt{\frac{e^2x^2-1}{x^2-1}}$ from the centre. And proceeding exactly in the same way as was done for the ellipse we shall have $H \pm H^{(n)} = e \times (xx' - x'x'' + x''x''' - x'''x''''$, &c. to n terms). Where $H^{(n)}$ denotes an arc of the same hyperboba, intercepted between the principal vertex, and a perpendicular ordinate at the distance $x^{(n)}$ from the centre, where $x = \frac{1}{e}\sqrt{\frac{e^2x^2-1}{x^2-1}}$, $x'' = \frac{1}{e}\sqrt{\frac{e^2x'^2-1}{x'^2-1}}$ $x''' = \frac{1}{e}\sqrt{\frac{e^2x''^2-1}{x''^2-1}}$, $x'''' = \frac{1}{e}\sqrt{\frac{e^2x'''^2-1}{x'''^2-1}}$, &c. the upper or lower sign takes place according as n is even or odd.

ARTICLE XV.

A new Method of approximating towards the Roots of Equations of all Dimensions.

By Mr. P. **Barlow**, *R. M. Academy.*

The following general formula, for approximating towards the roots of equations, is I believe new; and as it possesses several advantages over the method of Double Position, (which is now most commonly practised) both as it is attended with less labour, and has a greater degree of convergency, it may not be unacceptable to the readers of the Mathematical Repository.

Let $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c. = w$, be any general equation, and let a be an approximate value of x, so that

$$a^n + pa^{n-1} + qa^{n-2} + ra^{n-3} + \&c. = v,$$

then will

$$a + \frac{(w - v)2a}{(n-1)w \text{ or } v + (n+1)a^n + (n-1)pa^{n-1} + (n-3)qa^{n-2} + \&c.}$$

be another approximate value of x (using w or v according as a is integral or decimal; which will in all cases, at least, double the number of figures in a, and in most cases the approximation is considerably greater.

It is difficult from the nature of the investigation, to state the precise limits of approximation that may be depended upon; but in all cases, the doubling of the number of figures assumed, may be considered as perfectly accurate; and, generally, if the error be considerably less than the absolute term, the first supposition (admitting it to be the nearest integer number), will give two decimals, and by using these three figures as a new value of a, six or seven figures may be safely confided in.

The method of approximating by double position, is generally said to double the number of figures each operation, but in the comparison that I have made of that method with this formula, I have frequently found that I could not depend upon the result to that extent; but I have attempted no equation by this rule, in which it failed of doubling the number of figures assumed, and generally I have obtained the same result in two operations, as I did by three in using the method of double position; and as each of them is attended with only half the labour, in squaring, cubing, &c. that is necessary by position, it is obvious that considerable facility is given to the solution of equations, by this formula. I have not leisure at present to send you the investigation, but will furnish you with it for your next number, if I have not previously published it in the introduction to my own tables.

Solution of Cubic Equations.

In this case the general formula reduces to

$$a + \frac{(w - v)a}{w \text{ or } v + 2a^3 + pa^2}.$$

1. Let there now be proposed the cubic equation

$$x^3 + 3x^2 + 3x = 140.$$

Assume $a = 4$, then

$$\begin{array}{rcr@{\qquad\qquad}rcl}
a^3 & = & 64 & 2a^3 & = & 128 \\
3a^2 & = & 48 & pa^2 & = & 48 \\
3a & = & \underline{12} & w & = & \underline{140} \\
v & = & 124 & & & 316 \text{ divisor.} \\
w & = & \underline{140} & & & \\
(w-v) & = & 16 & & & \\
a & = & \underline{4} & & &
\end{array}$$

$316)\,64\,({\cdot}20$ therefore $x = 4{\cdot}20$

Assuming therefore $a = 4{\cdot}2$

$$\begin{array}{rcr@{\qquad\qquad}rcl}
a^3 & = & 74{\cdot}088 & 2a^3 & = & 148{\cdot}176 \\
3a^2 & = & 52{\cdot}92 & pa^2 & = & 52{\cdot}92 \\
3a & = & \underline{12{\cdot}6} & w & = & \underline{140{\cdot}} \\
v & = & 139{\cdot}608 & & & 341{\cdot}096 \text{ divisor.} \\
w & = & \underline{140{\cdot}} & & & \\
(w-v) & = & {\cdot}392 & & & \\
a & = & \underline{4{\cdot}2} & & & \\
 & & 784 & & & \\
 & & \underline{1568} & & &
\end{array}$$

$341{\cdot}096)\,1{\cdot}6464\,({\cdot}004827$*. Therefore $x = 4{\cdot}204827$, which is true to the last figure; for by adding 1 to each side of the proposed equation we have

$(x+1)^3 = 141$; therefore $x = -1 + \sqrt[3]{141} = 4{\cdot}2048279$.

2. Given $x^3 - 15x = 30$, to find x.

Assume $a = 5$; then

$$\begin{array}{rcr@{\qquad\qquad}rcl}
a^3 & = & 125 & 2a^3 & = & 250 \\
-15a & = & \underline{-75} & w & = & \underline{30} \\
v & = & 50 & & & 280 \text{ divisor.} \\
w & = & \underline{30} & & & \\
(v-w) & = & 20 & & & \\
a & = & \underline{5} & & &
\end{array}$$

$280)\,100\,({\cdot}36$. Whence $x = 4{\cdot}64$; and assuming this as a new value of a, we have

$$\begin{array}{rcl@{\qquad\qquad}rcl}
a^3 & = & 99{\cdot}89734 & 2a^3 & = & 199{\cdot}794 \\
-15a & = & \underline{-69{\cdot}6} & w & = & \underline{30} \\
v & = & 30{\cdot}29734 & & & 229{\cdot}794 \text{ divisor.} \\
w & = & \underline{30} & & & \\
(v-w) & = & {\cdot}29734 \text{ excess} & & & \\
a & = & \underline{4{\cdot}64} & & & \\
 & & 118936 & & & \\
 & & 178404 & & & \\
 & & \underline{118936} & & &
\end{array}$$

$229{\cdot}794)\,1{\cdot}3796576\,({\cdot}006003$

---

* The last quotient figure should always be taken the nearest, whether in excess or defect.

which taken from $4{\cdot}64$, leaves $x = 4{\cdot}633997$, which is true to the sixth place; for by Cardan's rule $x = \sqrt[3]{25} + \sqrt[3]{5} = 4{\cdot}633993$; our root thererefore is only deficient in the seventh figure.

These two examples are selected, because their roots are readily ascertained by other methods, by means of which we may judge of the degree of approximation obtained by the above formula.

Solution of Biquadratic Equations.

In this case the general formula becomes

$$a + \frac{(w-v)2a}{3w \text{ or } 3v + 5a^4 + 3pa^3 + qa^2 - ra}.$$

3. Let there now be proposed the equation

$$x^4 - 3x^2 - 75x = 10000.$$

Assume $a = 10$, then

$$\begin{array}{rcr@{\qquad\qquad}rcr}
a^4 &=& 10000 & 5a^4 &=& 50000 \\
-3a^2 &=& -\ 300 & qa^2 &=& -\ 300 \\
-75a &=& \underline{-\ 750} & -ra &=& +\ 750 \\
v &=& 8950 & 3w &=& \underline{30000} \\
w &=& \underline{10000} & & & 80450 \text{ divisor.} \\
(w-v) &=& 1050 & & & \\
2a &=& \underline{20} & & &
\end{array}$$

$80450)21000({\cdot}2610$. Whence $x = 10{\cdot}2610$, which will be found correct to the fifth place, in a single operation.

We will add another example in an equation of the fifth degree, which it is presumed will be sufficient to shew the advantage of the above approximating formula, as well as for illustrating its application, to any case that may arise.

Solution of Equations of the fifth degree.

Here the general formula becomes

$$a + \frac{(w-v)a}{2w \text{ or } 2v + 3a^5 + 2pa^4 + qa^3 - sa}.$$

4. Let there be proposed the equation

$$x^5 - 5x^3 + 5x = 1.$$

Assume $a = {\cdot}2$, then

$$\begin{array}{rcr@{\qquad\qquad}rcr}
a^5 &=& {\cdot}00032 & 3a^5 &=& {\cdot}00096 \\
-5a^3 &=& -\ {\cdot}04 & qa^3 &=& -\ {\cdot}04 \\
5a &=& \underline{1{\cdot}0} & -sa &=& -\ 1{\cdot}0 \\
v &=& {\cdot}96032 & 2v &=& \underline{1{\cdot}92064} \\
w &=& \underline{1{\cdot}00000} & & & {\cdot}88160 \text{ divisor.} \\
(w-v) &=& {\cdot}03968 & & & \\
a &=& \underline{{\cdot}2} & & &
\end{array}$$

${\cdot}8816){\cdot}007936({\cdot}009$. Therefore $x = {\cdot}209$.

Assuming this as a new value of a, we have

$$\begin{array}{rcr@{\qquad}rcr}
a^5 & = & \cdot0003988 & 3a^5 & = & \cdot0011964\\
-5a^3 & = & -\ \cdot0456466 & qa^3 & = & -\ \cdot0456466\\
5a & = & \underline{1\cdot045} & -sa & = & -\ 1\cdot045\\
v & = & \cdot9997522 & 2v & = & \underline{1\cdot9995044}\\
w & = & \underline{1\cdot0000000} & & & \cdot9100542 \text{ divisor.}\\
(w-v) & = & \cdot0002478 & & & \\
a & = & \underline{\cdot209} & & & \\
 & & 22302 & & & \\
 & & \underline{4956} & & &
\end{array}$$

$\cdot91005)\cdot0000517902(\cdot0000569$. Therefore $x = \cdot2090569$, which is true to the last place, the root being $\cdot209056926 = 2 \sin 6°$.

In this example it will be observed that we have used $2v$ in the denominator instead of $2w$, because the first assumed value of a was a decimal.

The foregoing examples haveb een taken promiscuously except their being such, that their roots might be found by other rules, in order to shew more clearly the converging property of the general formula. Besides the examples above given I have applied this method to several other equations, and have always found it succeed to my satisfaction, and, in several instances, have thus detected errors in the sixth or seventh figures of the root, of *old standing examples*, which I have been at the trouble of solving by direct and tedious methods, in order to see whether the defect was in this rule, or in that from which the answers abovementioned were deduced; and, in all such cases, I have always found my own answers to be correct, providing I never more than doubled the number of figures each operation; this therefore may be fixed as the limit of approximation, and which may always be considered as perfectly safe. And I believe I may add that there is no other practical rule that can be relied on to the same extent.

ARTICLE XVI.

Properties of a right-angled triangle, and of the lines drawn, and figures formed in and about the diagram which accompanies the demonstration of the forty-seventh proposition of the first book of Euclid's Elements.

By Mr. JOHN BRANSBY, *Ipswich.*

PROPOSITION 1. (Fig. 13. Pl. B)

The triangle ABC being right-angled at B, $\text{CA}^2 = \text{CB}^2 + \text{BA}^2$.

This is the 47th proposition of Euclid's 1st book, and besides the demonstration given by Euclid, as it stands in Commandine, Simson, &c. there are two additional ones in Clavius's Euclid, similar to the former of which is one in Pardie's Geometry, translated by Dr. Harris: two in the second book of Simpson's Geometry, one of which is nearly similar to that in Ward's Mathematician's Guide: one in the second book of Emerson's Geometry: an analytical investigation in Saunderson's Algebra, &c.—Having examined these demonstrations of this famous proposition, I am induced to subscribe to the following opinion of Clavius:—*Video demonstrationem Euclidis simpliciorem esse, et magis expeditam; sed non injucundum tamen est intelligere, variis demonstrationibus eandem veritatum posse confirmari.*

PROP. 2.

CPQ is a right-angled triangle, and similar and equal to ABC.

Dem. The ∠ BCQ = ACP, each being a right angle; take ACQ from each; then BCA = PCQ. Alo ∠ ABC = PQC, being right angles; therefore △ CPQ is similar to ABC (Cor. 2. 31. I. Barrow's Geo.); but AC = CP, being sides of the square ACPN; therefore △ CPQ = △ ABC.

Cor. CQ = BC, and PQ = AB.

PROP. 3.

NA*e* is a right-angled triangle, and similar and equal to ABC. The demonstration is similar to that of the last proposition.

Cor. N*e* = BC, and A*e* = AB.

PROP. 4.

CFH is a right-angled triangle, and similar and equal to ABC.

Dem. The ∠ ACH = FCB, each being a right-angle; take the ∠ BCH from each; then ∠ BCA = FCH. Also ∠ ABC = HFC, being right angles, therefore △ CFH is similar to ABC: but BC = CF, being sides of the square BCFG; therefore △ CFH = △ ABC.

Cor. 1. FH = AB = PQ = A*e*, and CH = AC.

Cor. 2. HI = BC; for CFIK being a parallelogram by construction, FI = CK = CB + BA: but cor. 1. FH = AB; therefore HI = BC.

PROP. 5.

AL*d* is a right-angled triangle, and similar and equal to ABC. The demonstration is similar to that of the 4th proposition.

Cor. 1. L*d* = BC = CQ = N*e*; A*d* = AC, and AL = AB.

Cor. 2. I*d* = AB; for GILA being a parallelogram, LI = AG = AB + BC, but L*d* = BC, cor. 1; therefore I*d* = AB.

PROP. 6.

GBK, BK*r*, GIK, and HI*d* are right-angled triangles, and similar and equal to ABC.

Dem. BG = B*r* = KI = BC by construction, and IH = BC (cor. 2. pr. 4). Also BK = GI = I*d* = AB (by construction and cor. 2. pr. 5), and the angles at B and I are equal, being right angles; therefore the Δs GBK, BK*r*, GIK, HI*d* and ABC are similar and equal in all respects. (Euc. 4. I.).

Cor. GK = H*d* = K*r* = AC.

PROP. 7.

A*n*C is a right-angled triangle, and similar and equal to ABC.

Dem. A*n*CB is a parallelogram by construction, and AC its diameter; therefore Δ A*n*C is similar and equal to ABC (34. I.).

PROP. 8.

The triangles HFY, AL*b* and B*i*K are similar and = to ABD.

Dem. Those triangles are respectively similar to the triangles CFH, AL*d* and BK*r* (8. VI.), and these are similar to ABC (prop. 4, 5, 6.) and therefore to one another; and the triangle ABD is similar to ABC (8. VI.), therefore the triangles HFY, AL*b* and B*i*K are similar to ABD; and they are also equal to it because FH = AL = BK = AB.

Cor. HY = A*b* = K*i* = AD, and FY = L*b* = B*i* = BD.

PROP. 9.

The triangles CFY, L*bd* and B*ri* are similar and = to CBD.

The demonstration is similar to that of the last proposition.

Cor. CY = *db* = *ri* = CD.

PROP. 10.

AM = CX.

Dem. AML*b* and CXFY are parallelograms by construction, therefore AM = L*b* and CX = FY: but L*b* = FY (cor. pr. 8.); therefore AM = CX.

Cor. AX = CM = BO.

PROP. 11.

FX + LM = AC.

Dem. FX = CY = CD (cor. prop. 9.) and LM = A*b* = AD (cor. pr. 8.); therefore FX + LM = CD + AD = AC.

PROP. 12.

BD is a mean proportional between FX and LM.

Dem. CD : DB :: DB : DA (cor. 8. vi.), and FX = CD, and LM = AD; therefore FX : DB :: DB : LM.

PROP. 13.

Gm + Kh = AC + 2BD = XM.

Dem. The right-angled triangles Bir, BGu are equal in all respects, because Br = BG; therefore ri = Gu = CD (cor. pr. 9.), and Ki = AD (cor. pr. 8.), also um = BD = hi, therefore Gm = CD + DB and Kh = AD + DB, hence

Gm + Kh = AC + 2BD = XM (because CX = AM = BD).

Cor. 1. The sum of the four perpendiculars FX, Gm, Kh and LM is = twice the sum of AC and BD. For FX + LM = AC (pr. 11.), and Gm + Kh = AC + 2BD (this prop.); therefore FX + Gm + Kh + LM = 2(AC + DB).

Cor. 2. Dm = Dh; Gm = Ch; Kh = Am.

PROP. 14.

The rectangle Gm . Kh = twice (BD^2 + △ CBA).

Dem. Gm = CD + DB, and Kh = AD + DB (dem. to prop. 13.); theref. Gm . Kh = (CD + DB)(AD + DB) = $2BD^2$ + BD × (CD + AD) = 2(BD^2 + △ABC).

Cor. Gm . Kh = 2BD . Bk. For Bk is = half GK = half AC.

PROP. 15.

If BD be perpendicular to AC it passes through the point of intersection of CL and AF.

Dem. Let z be the point in which CL intersects BD, and *a* that in which AF intersects it; then

AX : AD :: XF : D*a* (4. vi.); but AX = CM (cor. pr. 10.), AD = LM and XF = DC; therefore CM : LM :: DC : D*a*: but CM : LM :: DC : DZ (4. vi.); therefore DZ = D*a*, and Z coincides with *a*.

PROP. 16.

The triangle ASC = BKS and the △WBG = CWA.

Dem. The △BKS = BLS (37. i.); add CBS to each, then △CKS = CBL: But CBL = CBA (ibid); therefore △CKS = CBA; take away △CBS, then △ASC = BKS. In the same manner it may be proved that △WBG = △CWA.

Cor. 1. The △*s*, CKS, CBL, ABF, AWG are each = △ABC.

Cor. 2. The rectangle contained by BS and the sum of the sides AB, BC is = twice the △ABC; for it is = twice the △CKS. Also, the rectangle contained by BW and the sum of the sides AB, BC is = twice the △ABC; for it is = twice the △AWG which is = ABC.

PROP. 17.

BS = BW.

Dem. BS × (AC + BC) = 2 △ABC = BW × (AC + BC) (cor. prop. 16.); therefore BS = BW.

PROP. 18.

The △ASC = BWA and the △CWA = BSC.

Dem. BS = BW (prop. 17.); therefore △BWA = BKS; but △BKS = ASC (prop. 16.); therefore △ASC = △BWA. In the same manner it may be proved that △CWA = BSC.

Cor. The triangle BSC = WBG, and △WAB = BSK.

PROP. 19.

The quadrilateral WBSZ is = to the triangle CZA.

Dem. The triangle ASC = BWA (prop. 18.); take away the common part ASZ, then the quadrilateral WBSZ = △CZA.

PROP. 20.

The angle SDW is a right angle.

Dem. AS : BS :: AL = AB : BC (4. vi.), and
AD : BD :: AB : BC (8. vi.);
AS : BS :: AD : BD, ex equo; therefore DS bisects the right-angle BDA (3. vi.), or BDS = half a right-angle.

Again, BW : CW :: AB : CF = BC (4. vi.)
BD : DC :: AB : BC (8. vi.),
BW : CW :: BD : DC ex equo, therefore DW bisects the right angle BDC, or BDW = half a right angle; consequently SDW is a right angle.

PROP. 21.

BS is a mean proportional between AS and CW.

Dem. AS : BS :: AB : BC and BW : CW :: AB : BC (dem. pr. 20.) therefore AS : BS :: BW : CW: but BS = BW (prop. 17.); therefore AS : BS :: BS : CW.

Cor. $WS^2 = 2AS \cdot CW$.

PROP. 22.

The triangle DWC is similar to DSB, and the △DWB to DSA.

Dem. The angle DCW = DBS, each being the complement of DBC; and ∠CDW = SDR, each being = half a right-angle (dem. to prop. 20.); therefore △DWC is similar to DSB. In a similar manner it may be proved that the △DWB is similar to DSA.

PROP. 23.

AS : CW :: AD : DC.

Dem. The △DWC is similar to DSB (prop. 22.); therefore CW : CD :: BS : BD, also the △DWB is similar to DSA (ibid), and AS : AD :: BW = BS : BD; therefore AS : CW :: AD : DC.

Cor. $AS : CW :: AB^2 : BC^2$.

PROP. 24.

The triangles PCF, ALN and GBK, formed by lines joining the exterior angles of the squares, are each equal to the triangle ABC.

Dem. $CF = CQ$; therefore $\triangle PCF = PQC$; and $\triangle PQC = ABC$ (prop. 2.); therefore $\triangle PCF = ABC$. Again, because $AL = Ae$ (cor. 1. prop. 3.), the $\triangle ALN = ANe$, and the $\triangle ANe = ABC$; therefore the $\triangle ALN = ABC$. Also $\triangle GBK = ABC$ (prop. 6.).

Prop. 25.

The whole space $PFGKLNP$ is = four times the $\triangle PBN$.

Dem. The whole space is composed of the four triangles PFC, GBK, NLA, CBA and the three squares on AC, AB, BC; which (by prop. 24 and 1.) are equal to four times the $\triangle ABC$ and twice the square on AC; that is $= 2AC\,(AC + BD) = 2PN \cdot OB =$ four times the $\triangle PBN$.

Prop. 26.

The figure $ACHd$ is a square.

Dem. $AC = CH = Hd = dA$ (cors. prop. 4 and 5.) and the angles CAd and ACH right angles; therefore $ACHd$ is a square.

Prop. 27.

The figure $LnFI$ is a square.

Dem. $FI = FG + GI = CB + BK = AB + BC = AG = nF$; but $FI = nL$, and $nF = LI$; and IFn is a right angle; therefore $LnFI$ is a square.

Prop. 28.

The parallelogram $OC =$ parall. CI, and parall. $AO =$ parall. AI.

Dem. They are on equal bases (PC and CH), (NA and Ad) and between the same parallels, and therefore equal by 36. i.

Cor. Parall. $OI +$ parall. $AI =$ square on AC.

Prop. 29.

The figures $HFPQ$, $NedL$ are parallelograms, and each = to four times the triangle ABC.

Dem. PQ is $= FH$ (cor. 1. prop. 4.), and parallel to it because QC, CF are in the same straight line; therefore PF is = and parallel to QH (33. 1.), and HFPQ is a parallelogram. Also the $\triangle QCH = PCF = ABC$ (prop. 24.) and the $\triangle PQC = CFH = ABC$ (37. 1.); therefore the parallelogram HFPQ is = 4 times the triangle ABC. It may be proved in a similar way that $NedL$ is a parallelogram and $= 4\,\triangle ABC$.

Prop. 30.

The points L, B, F are in the same straight line.

Dem. $LnFI$ is a square (prop. 27.), also LB and BF are squares by construction, and LBF is the diameter on which these less squares are constructed (4. II.); therefore LBF is a straight line.

Prop. 31.

The points Q, S, K are in the same straight line.

Dem. Because $Br = BC$ and $BS = BW$ (prop. 17.), Sr is $= WC$; and Qr is $= FC$, each being $= BC$, and the $\angle nAr = FCW$; therefore the $\triangle Qrs$ is equal and similar to CWF (4. I.), and the $\angle QSr = CWF$. Also the $\triangle BSK$ is equal and similar to BWA (cor. prop. 18.); therefore ∠ BWA = BSK and ∠ QS*r* = BSK; therefore the points Q, S, K are in the same straight line (14, 15. I.)

PROP. 32.

The points *e*, W, G are in the same straight line.

Dem. This may be proved as in the last proposition by drawing a perpendicular from *e* upon BC: then the ∠*e*WC = LBA, and ∠GWB = BSC; therefore ∠*e*WC = GWB, and the points *e*, W, G are in the same straight line.

PROP. 33.

BN and CL cross each other at right angles.

Dem. The ∠NAC = BAL; add CAB to each, then the ∠NAB = CAL, and △NAB = and similar to CAL; theref. ∠ABN = ALC, or ∠SB*x* = ALS; and the ∠BS*x* is = ASL (15. I); therefore ∠B*x*S = SAL = a right angle.

Cor. B*x* is a mean proportional between C*x* and S*x*.

PROP. 34.

BP and AF cross each other at right angles.

The demonstration is similar to that of the last proposition.

Cor. B*y* is a mean proportional between W*y* and A*y*.

PROP. 35.

WE is parallel to BT, and SQ to BP.

Dem. The ∠SB*x* = ALS (dem. to prop. 33.) = SCB (8. VI.) = ∠WGB, because the △BCS is = and similar to BGW (cor. prop. 18.); therefore WE is parallel to BT (28. I.). It may be proved in a similar manner that SQ is parallel to BP.

PROP. 36.

The figure N*e*GB is a parallelogram and = twice the △ABC.

Dem. N*e* is = and parallel to GB, therefore N*e*GB is a parallelogram; and it is = to the rectangle whose base is GB (= BC) and altitude A*e* (= AB), or twice the triangle ABC.

PROP. 37.

The figure PQKB is a parallelogram and = twice the △ABC.

Dem. PQ is = and parallel to BK, therefore PQKB is a parallelogram; and it is = to the rectangle whose base is BK (= AB) and altitude CQ (= BC), or twice the triangle ABC.

PROP. 38.

TV is a mean proportional between AT and VC.

Dem. SQ is parallel to BV (prop. 35.), therefore AS : BS :: AT : TV. Also WE is parallel to BT (pr. 35.), therefore BW : CW :: TV : CV; but AS : BS :: BW = BS : CW (prop. 21.); therefore

AT : TV :: TV : VC.

Cor. AS : CW :: AT : CV.

For AS : BS :: AT : TV, and CW : BS :: CV : TV; *therefore* AS : CW :: AT : CV.

Prop. 39.

DA : CD :: TA : CV.

Dem. AS : CW :: DA : CD (prop. 23.) and AS : CW :: AT : CV (cor. prop. 38.); therefore DA : CD :: AT : CV.

Cor. CV : VD :: TA : DT, and DA : CD :: DT : VD.

Prop. 40.

AS and SB, BW and WC, SD and DW are all in the ratio of the legs of the triangle ABC.

Dem. AS : BS :: AB : BC and BW : WC :: AB : BC (dem. to pr. 21). The triangle DWC is similar to the triangle DSB (prop. 22.); therefore SD : DW :: BD : DC, but AB : BC :: BD : DC (8. VI.); therefore SD : DW :: AB : BC.

Prop. 41.

The ΔDWC : ΔDWB :: ΔDBS : ΔDSA :: BC : AB.

Dem. The ΔDWC : ΔDWB :: CW : WB (1. vi.) and BC : AB :: CW : WB (dem. prop. 20.); therefore ΔDWC : ΔDWB :: BC : AB. Again ΔDBS : ΔDSA :: BS : AS (1.. VI.) and BC : AB :: BS : AS (dem. pr. 20.); therefore ΔDBS : ΔDSA :: BC : AB.

Prop. 42.

$LN^2 + PF^2 = 5AC^2$.

Dem. $eL = 2AB$ and $Ne = CB$ (cor. pr. 3.); $QF = 2CB$ and $PQ = AB$ (cor. pr. 2.); therefore $LN^2 = eL^2 + eN^2 = BC^2 + 4AB^2$, and $PF^2 = QF^2 + PQ^2 = 4BC^2 + AB^2$; therefore $LN^2 + PF^2 = 5(BC^2 + AB^2) = 5AC^2$ (prop. 1.).

Cor. The sum of the squares of all the sides of the polygon PFGKLNP is eight times the square on AC.

Prop. 43.

Ap : pW : : sq : qC,

Dem. Because Bp is parallel to CW (prop. 35.), AB : GB = BG : : Ap : pW, and because BV is parallel to KS (ibid), BK = AB : BC : : sq : qC; therefore Ap : pW : : sq : qC.

Cor. Ap : AW : : sq : CS.

Prop. 44.

Ap : Bp :: Bq : Cp.

Dem. The ∠BAy = WBy and SBp = BCq (8. vi.); therefore the Δ ApB is similar to the ΔBqC, and Ap : Bp : : Bq : Cq.

Prop. 45.

Bp = sq, and Bq = Wp.

Dem. The ∠WBp + ∠xBS = a right angle, and ∠qsB + ∠xBS = Bxs a right angle; take away the common angle xBS; then the angle WBp = qSB. Again ∠WBq + ∠qBS = a right angle, and ∠pWB + ∠WBy = a right angle; take away the common ∠WBy, then the angle qBS = pWB; therefore the ΔWpB is similar to the ΔsqB, and since BW = BS (pr. 17.) these triangles are equal in all respects; therefore Bp = sq and Bq = Wp.

Or the proposition may be demonstrated differently as follows: Ap : pW : : sq : qC (pr. 43); Ap : Bp : : Bq : qC (pr. 44); therefore pW : Bp : : Bq : sq : and the ∠BpW = Bqs, each being the complement of pZS; therefore the $\triangle WBp$ is similar to Bqs (6. VI.), and since $BW = BS$ (pr. 17.) these triangles are equal in all respects; therefore $Bp = Sq$ and $Bq = Wp$.

Prop. 46.

DZ is a mean proportional between VD and DT.

Dem. The triangle DVB is similar to DZA; therefore $VD : DB :: DZ : DA$: but $CD : DB :: DB : DA$ (8. VI.); therefore $VD : DZ :: CD : DB$, and VZ is parallel to CB. Again the triangle DBT is similar to CDZ: therefore $DT : DB :: DZ : CD$, and $DT : DZ :: DA : DB$. Consequently TZ is parallel to AB, and the $\angle VZT = ABC =$ a right angle. Therefore ZD is a mean proportional between VD and DT (8. VI.).

Cor. $pT . Vq = pZ . Zq$. For the triangle TZp is similar to ApB and the triangle VZq to BqC: but ApB is similar to BqC (dem. to prop. 44.), and consequently TZp is similar to VZq; therefore $pT : pZ :: Zq : Vq$, or $pT . Vq = pZ . Zq$.

Prop. 47.

$BZ = VT$.

Dem. $CV : BZ :: VD : DZ$ and $AT : BZ :: DT : DZ$; therefore $CV . AT : BZ^2 :: VD . DT : DZ^2$. But $VD . DT = DZ^2$ (pr. 46); therefore $BZ^2 = CV . AT = VT^2$ (pr. 38.), or $BZ = VT$.

Cor. 1, BZ is a mean proportional between CV and AT.

Cor. 2. $BZ^2 = VZ^2 + ZT^2$.

Prop. 48.

$BT . BV = AZ . ZC$.

Dem. The $\triangle CBV$ being similar to BZA and BAT to BZC, $CV : VB :: BZ : ZA$ and $AT : BT :: BZ : ZC$; therefore $AT . CV : BT . VB :: BZ^2 : AZ . ZC$; but $AT . CV = BZ^2$; therefore $BT . BV = AZ . ZC$.

Prop. 49.

If Vw and Tv be drawn perpendicular to AC to meet the sides BC, BA of the triangle ABC in w and v, the figure $VwvT$ is a square.

Dem. For $VwBZ$ and $ZBvT$ being parallelograms, Vw is $= BZ = Tv = VT$; and the figure $VwvT$ is a square.

Prop. 50.

The following rectangles are respectively equal, viz. $AD . AV = AZ . Ay$; $CD . CT = CZ . Cx$; $By . Bq = Bx . Bp$; $Zx . Zq = Zy . Zp$; $By . BV = BZ . BD = BT . Bx$; $CZ . Zx = BZ . ZD = AZ . Zy$; $NT . Tx = CT . TA$; $PV . Vy = CV . VA$; $FW . Wy = CW . WB$; $LS . Sx = AS . SB$; and $Bx . BN = BZ . BO = By . BP$.

Dem. Because of the right angles at D, y and x, the points V, D, Z, y are in a circle, therefore (36, 37. III.) $AD . AV = AZ . Ay$ and $By . BV = BZ . BD$. Also the points T, D, Z, x are in a circle, therefore $CD . CT = CZ . Cx$, and $BT . Bx = BZ . BD$. Again the points p, x, y, q are in a circle, therefore $By . Bq = Bx . Bp$.

and $zx \cdot zq = zy \cdot zp$. The points B, x, D, C, are also in a circle, and so are the points B, y, D, A; therefore $CZ \cdot ZX = BZ \cdot ZD = AZ \cdot Zy$. It may be proved in a similar manner that the other rectangles are respectively equal.

Cor. 1. $BV : BT :: Bq : Bp$.

For $By \cdot Bq = Bx \cdot Bp$ and $By \cdot BV = BT \cdot Bx$,
or $Bx : By :: Bq : Bp$ and $Bx : By :: BV : BT$;
Therefore $BV : BT :: Bq : Bp$.

Cor. 2. Therefore pq is parallel to AC.

Cor. 3. $Ap : AZ :: Cq : CZ$.

Prop. 51.

Draw SR parallel to CB and join WR: then will the figure SRWB be a square.

Dem. $AS : SR :: AB : BC$ (4. VI.), $AS : BS :: AB : BC$ (dem. to pr. 20.); therefore $SR = BS = BW$ (pr. 17.) and the figure SRWB is a square.

Cor. $AR : CR :: AB : BC$.

Prop. 52.

Sx is a mean proportional between Bx and px, and Wy a mean proportional between By and qy.

Dem. $Ap : pW :: AB : BC$ (dem. to prop. 43.)
$AS : BS :: AB : BC$ (dem. to prop. 20.);
therefore $Ap : pW :: AS : BS$, and the points R, p, S, are in a straight line. In a similar way it may be proved that the points R, q, W are in a straight line; therefore the triangles BSp and BWq are right-angled at S and W: consequently Sx is a mean proportional between Bx and xp, and Wy a mean proportional between By and qy. (8. VI.).

Cor. BS is a mean proportional between Bx and Bp, and also between By and Bq (8. VI).

Prop. 53.

$$BS = Sp + Wq.$$

Dem. Because WRSB is a square (pr. 47.) the triangles pSB and qRS are right angled at S and R; also $Bp = Sq$ (pr. 45.) and $BS = RS$, those triangles are equal in all respects (4. I.); therefore $pS = qR$: but $qR + Wq = BS$, therefore $BS = Sp + Wq$.

Cor. $Wq = pR$, and $pq^2 = Sp^2 + Wq^2$.

Prop. 54.

The points R, D, W, B, S are in a circle, of which SW or RB is a diameter.

Dem. WBS is a right angle by constr. and WDS a right angle (pr. 20.), therefore a circle will pass through the angles of the quadrilateral BSDW (22. III.): also because RSBW is a square, the angle WRS is a right angle and = WDS; therefore the circle will also pass through the point R: and because of the right angles, SW and RB are evidently diameters.

a

s

p

th
st
R
B
pr
be

be

an
BS
ps

a

(p
qu
the
cle
an

ARTICLE XVII.

Mathematical Scraps. By Mr. THOMAS WHITE*.

To the Editor of the Mathematical Repository.

SIR,

IF you think that the following Mathematical Scraps are worthy of a place in your Repository, they are at your service.

Dumfries Mathematical School,
23d Nov. 1812.
THOMAS WHITE.

I. *A new answer to the 149th quest.* p. 42, vol. ii.

Question: To place a straight rod, or beam, loaded with any given weights, so that it may be in equilibrio, resting upon an upright prop, and one end touching a smooth vertical wall given by position?

G (fig. 8. pl. A′.) is the place of the centre of gravity of the rod and weights; W = weight of the rod and weights lodged in G; Gb, in the vertical, represents W; complete the rectangle Gb; make $Bc = Ga$; complete the rectangle Bc; and, by the lever, Bf, perpendicular to BK, is $= \frac{GI}{BI} \cdot ab$; complete the rectangle Bf. Then, since BK is at rest, Bd must be $= Be$, which is one equation.

By gravity Gb, $\frac{Ga}{Gb} \cdot Gb = \frac{ID}{BI} \cdot W = Ga = Bc$, and, in this direction.

By the force Ga, $\begin{cases} \frac{Bd}{Bc} \cdot Bc = \frac{ID}{BI} \cdot \frac{ID}{BI} \cdot W = \frac{DI^2}{BI^2} \cdot W = \text{force } Bd\text{, upward.} \\ \frac{cd}{Bc} \cdot Bc = \frac{BD}{BI} \cdot \frac{ID}{BI} \cdot W = \frac{BD \cdot DI}{BI^2} \cdot W = \text{re-action } dc \text{ of the plane.} \end{cases}$

---

[*] Master of the Mathematical School, in the Academy of Dumfries. He receives young gentlemen as boarders.

By gravity Gb, $\frac{ab}{Gb} . Gb = \frac{BD}{BI} . w = ab$, perpendicular to the rod; which force, by the lever, is equivalent to the force $\frac{GI . BD}{BI^2} . w$ at B, represented by the perpendicular Bf.

$$\text{By the force } ab, \begin{cases} \frac{Be}{Bf} . Bf = \frac{BD}{BI} . \frac{GI . BD}{BI^2} . w = \frac{GI . BD^2}{BI^3} . w = \\ \quad \text{force } Be, \text{ downwards;} \\ \frac{fe}{Bf} . Bf = \frac{ID}{BI} . \frac{GI . BD}{BI^2} . w = \frac{DI . IG . BD}{BI^3} . w \\ \quad = \text{force, or re-action } fe. \end{cases}$$

Now, 1st. For the equilibrium of the rod: Bd must be $=$ Be; that is $\frac{DI^2}{BI^2} . w = \frac{GI . BD}{BI^3} . w$; and, hence $BI^3 = BG . BD^2$; hence BI is given and the position of the rod is given.

2d. For the perpendicular pressure against the vertical plane AB. It is $= cd + fe = \frac{BG . BD . DI}{BI^3} . w =$ (substituting for BI^3) to $\frac{DI}{BD} . w$; and $DI = \sqrt{(BI^2 - BD^2)}$.

3d. For the vertical pressure at I, or in direction ID, make B$g = cd + df$; draw the perpendicular gh; then $\frac{gh}{Bg} . Bg = \frac{DI}{BD} . \frac{DI}{BD} . w = \frac{DI^2}{BD^2} . w =$ vertical weight or pressure at B; hence that at I, sustained by the prop, is $= w + \frac{DI^2}{BD^2} . w = \frac{BI^2}{BD^2} . w$.

4th. The pressure at I, perpendicular to the rod, is $\frac{BI}{BD} . w$; for $\frac{BD}{BI} . \frac{BI^2}{BD^2} . w = \frac{BI}{BD} . w$.

II. *Question.* Let the end B (fig. 9. pl. A′.) of a given beam BA rest against a given point B in a vertical plane PBG, when the other end A is sustained by a given weight W fastened to a string which passes over a pulley P in the plane: Now when the beam is in equilibrio, what is its position, and where is the place of the pulley?

Solution. Let A*a* represent the weight W; complete the rectangle A*a*; the vertical C*f*, the weight *w* of the beam; complete the rectangle C*f*; on AB take B*c* $=$ A*b* $-$ C*g*; drop the perpendicular *cd*. The perpendicular force C*e*, at C, is equivalent to the perpendicular force $\frac{Bc}{BA} \cdot Ce$, at A $=$ perpendicular A*h*, suppose; and, *ab* must be $=$ A*h*, which is one equation:

Again, the perpendicular force C*e*, at C, is equivalent to the perpendicular force $\frac{AC}{AB} \cdot Ce$, at B, $=$ perpendicular B*k*, suppose; drop the perpendicular *kl*; then, since all is at rest, *d*B must be $=$ B*l*, which is a second equation.

We have $Ab = \sqrt{(W^2 - ab^2)}$, in which *ab* is unknown; draw AG perpendicular to the vertical; then AG is $= \sqrt{(AB^2 - BG^2)}$; in which BG is unknown.

By gravity C*f*, $\frac{Cg}{Cf} \cdot w = \frac{BG}{AB} \cdot w =$ force C*g*, downward, in direction of the beam; then

By the force C*g*,
$$\left\{\begin{array}{l} Bc = Ab - Cg = \sqrt{(W^2 - ab^2)} - \frac{BG}{AB} \cdot w \\ \quad = \text{force in direction AB.} \\ \frac{Bd}{Bc} \cdot Bc = \frac{BG}{AB} \cdot \left[\sqrt{(W^2 - ab^2)} - \frac{BG}{AB} \cdot w\right] \\ \quad = \text{force Bd, vertically upward.} \\ \frac{cd}{Bc} \cdot Bc = \frac{AG}{AB} \cdot \left[\sqrt{(W^2 - ab^2)} - \frac{BG}{AB} \cdot W\right] \\ \quad = \text{force cd, horizontal pressure.} \end{array}\right.$$

By gravity C*f*. $\frac{Ce}{Cf} \cdot w = \frac{AG}{AB} \cdot w = \perp$ force C*e*.

$$\text{By the force } Ce, \left\{\begin{array}{l} \left.\begin{array}{l} \frac{BC}{BA}.Ce = \frac{BC\ .AG}{AB^2}.w = \perp \text{ force } Ah, \\ \frac{AC}{BA}.Ce = \frac{AC.AG}{AB^2}.w = \perp \text{ force } Bk, \end{array}\right\} \text{by the lever.} \\ \frac{Bl}{Bk}.Bk = \frac{AG}{AB}.\frac{AC}{AB}.Ce = \frac{AC\ .\ AG^2}{AB^3}.w = \\ \quad \text{force } Bl, \text{ vertically downward.} \\ \frac{lk}{Bk}.Bk = \frac{BG}{AB}.\frac{AC}{AB}.Ce = \frac{AC.AG.BG}{AB^3}.w \\ \quad = \text{force } kl, \text{ horizontal pressure.} \end{array}\right.$$

Hence, the first equation is $ab = \frac{BC\ .\ AG}{AB^2}.w$; and the 2d is $\frac{BG}{AB}.[\surd(W^2 - ab^2) - \frac{BG}{AB}.w] = \frac{AC\ .\ AG^2}{AB^3}$; and, substituting for ab, &c. we have $BG.\surd(AB^2.[\frac{AB^2}{BC^2}.\frac{W^2}{w^2} - 1] + BG^2) = BG^2 + AB^2.\frac{AC}{BC}$; and therefore $BG = \frac{AB.AC.w}{\surd(AB^2.W^2 - BC.(AB+BC).w^2)}$; Hence, when $W = w$, BG is $= AB$; i. e. AB coincides with BG; as it ought. Again, $ab = \frac{BC.w}{AB^2}.\surd(AB^2 - BG^2)$, comes out $= BC\ .\ w\ .\sqrt{\frac{W^2 - w^2}{AB^2.W^2 - BC.(AB+AC).w^2}}$; and, since $aA = W$, the sine $\angle PAB$ is $= BC.\frac{w}{W}\sqrt{\frac{W^2 - w^2}{AB^2.W^2 - BC.(AB+AC).w^2}}$: Hence, when the weight W is given, the direction of PA, or $\angle$ PAB must be found; and P may be placed any where on PA.

III. A new proof of the prop. at p. (2), vol. ii, part ii. Retaining the ingenious author's notation:....Since, by the laws of gravity, $4lx = v^2$; $4ly = u^2$; $4lz = w^2$, &c. we have, by multiplying respectively by A, by B, by C, &c. and adding

$$4l.[Ax + By + Cz +, \ldots.] = Av^2 + Bu^2 + Cw^2 +, \ldots.$$

IV. In art. vi, part ii, vol. i, of the Repository, Mr. Ivory has solved a dynamical prob. which was first published in Simpson's Miscellaneous Tracts, page 131, prob. 3d; and in art. vi, page 18, vol. ii, part ii, of the Repository, Mr. Barry observes that, the result of that solution differs widely from a solution of the same given by Mr. Atwood, p. 189, of his Rec. Mot. and adds, " I find that his (Mr. Atwood's) error proceeds from a wrong application of his own general theorem in ascertaining the ratio of the 2d fluxions of x and y." But how this happened, has not, I believe, been yet shewn.

Mr. Atwood has $y = \sqrt{(a^2 + x^2)}$; and thence finds $\frac{\dot{y}}{\dot{x}} = \frac{x}{\sqrt{(a^2 + x^2)}}$; which is right; but he adds $\frac{\ddot{y}}{\ddot{x}}$ is $= \frac{a^2 + 2x^2}{2x\sqrt{(a^2 + x^2)}}$. Several years ago, I met with this apparently erroneous expression, in the work above mentioned, and took the liberty of writing to Mr. Atwood concerning it; but as I was never favored with a reply, concluded that either he must have been offended, or my letter must have been miscarried. The expression is, however, analytically true; yet, as shall be seen, it is not applicable to the dynamical prob. in which Mr. Atwood employs it; and may therefore be allowed to be mechanically false.

1st. It is analytically true: For by the usual process, $\frac{\ddot{y}}{\ddot{x}} = \frac{x}{\sqrt{(a^2 + x^2)}} + \frac{a^2}{(a^2 + x^2)^{1+\frac{1}{2}}}$ and $\frac{\dot{x}^2}{\ddot{x}}$, which, if Mr. Atwood's expression be true, must be $= \frac{x}{\sqrt{(a^2 + x^2)}} + \frac{a^2}{2x\sqrt{(a^2 + x^2)}}$; hence, if this equation be true, $\frac{\dot{x}^2}{\ddot{x}}$ is $= \frac{a^2 + x^2}{2x}$, in finite quantities; therefore, $(a^2 + x^2) \times \ddot{x} - 2x\dot{x}^2 = 0$; that is, restoring the last multiplier, $\frac{(a^2 + x^2)\ddot{x} - 2x\dot{x}^2}{(a^2 + x^2)^2} = 0$; of which the fluent is $\frac{\dot{x}}{a^2 + x^2} =$ some *invariable* fluxion $\frac{\dot{\alpha}}{a^2}$, suppose; and consequently $\frac{a^2\dot{x}}{a^2 + x^2} = \dot{\alpha}$, the invariable fluxion of an arc, (rad. *a*, and tang. *x*), which is true. Hence $\frac{\dot{x}^2}{\ddot{x}}$, $\frac{\dot{x}^3}{\dddot{x}}$, &c. and, of course, $\frac{\ddot{y}}{\ddot{x}}$, $\frac{\dddot{y}}{\dddot{x}}$, &c. are assignable in finite terms, if we suppose $\dot{\mathrm{D}}\mathrm{F}$ (fig. 89.) *invariable*, which Mr. Atwood appears to have covertly done. It is known that $\sqrt{(\dot{x}^2 + \dot{y}^2)}$, the fluxion of a curvilinear arc; $y\dot{x}$, the fluxion of the area of a curve; and other functions of *x* and *y* are frequently conceived to flow uniformly, by foreign mathematicians.

2dly. It is mechanically, or dynamically false: For, in the general solution, the expression for inertia is obtained by assuming $\dot{x}$ invariable; and, therefore, *no function* of *x* can afterwards be assumed constant, as it has been shewn Mr. Atwood has done.

Mr. Atwood seems to have been led into error by an anxiety to step out of the common tract, i. e. a wish to avoid the obvious substitution of $\frac{w}{v}$ for $\frac{\dot{y}}{\dot{x}}$, and $\frac{\dot{w}}{\dot{v}}$ for $\frac{\ddot{y}}{\ddot{x}}$.

Thus it appears that Mr. Atwood's is an oversight in principle; it will be seen that if $\frac{\ddot{y}}{\ddot{x}}$ be assumed as really $=$ $\frac{a^2 - 2x^2}{2x\sqrt{(a^2 + x^2)}}$, all that follows is quite correct; i. e. the fluent of $\dot{z} = \frac{\mathrm{B}a^2\dot{x} + \mathrm{B}x^2\dot{x} - 2\mathrm{A}x\dot{x}\sqrt{(a^2 + x^2)}}{(\mathrm{A}+\mathrm{B})a^2 + (\mathrm{B}+2\mathrm{A})x^2}$ is justly assigned.

For the above expression for $\dot{z}$ is $=$

$$\frac{\mathrm{B}a^2}{\mathrm{B}+2\mathrm{A}} \cdot \frac{\dot{x}}{\frac{\mathrm{A}+\mathrm{B}}{\mathrm{B}+2a} \cdot a^2 + x^2} + \frac{\mathrm{R}}{\mathrm{B}+2a} \cdot \frac{x^2\dot{x}}{\frac{\mathrm{A}+\mathrm{B}}{\mathrm{B}+2\mathrm{A}} \cdot a^2 + x^2} - \frac{2\mathrm{A}}{\mathrm{B}+2\mathrm{A}} \times \frac{x\dot{x}\sqrt{(a^2 + x^2)}}{\frac{\mathrm{A}+\mathrm{B}}{\mathrm{B}+2\mathrm{A}} \cdot a^2 + x^2}$$

; put $\frac{\mathrm{B}}{\mathrm{B}+2\mathrm{A}} = m$; $\frac{\mathrm{A}+\mathrm{B}}{\mathrm{B}+2\mathrm{A}} \cdot a^2 = b^2$, and $\frac{2\mathrm{A}}{\mathrm{B}+2\mathrm{A}} = n$; then $\dot{z}$ becomes $= \frac{ma^2\dot{x}}{b^2+x^2} + \frac{mx^2\dot{x}}{b^2+x^2} - \frac{nx\dot{x}\sqrt{(a^2+x^2)}}{b^2+x^2} =$

$$\frac{ma^2\dot{x}}{b^2+x^2}+m\dot{x}-\frac{mb^2\dot{x}}{b^2+x^2}-\frac{nx\dot{x}}{\sqrt{(a^2+x^2)}}-\frac{n(a^2-b^2)x\dot{x}}{(b^2+x^2)\sqrt{(a^2+x^2)}},$$

and the fluent is (by restoring m, n and b in all the terms save the last)

$$=\frac{\text{AB}\,.\,a\,.\,\text{M}^\circ}{\sqrt{(\text{A}+\text{B})}(\text{B}+2\text{A})^{\frac{3}{2}}}+\frac{\text{B}x}{\text{B}+2\text{A}}-\frac{2\text{A}}{\text{B}+2\text{A}}\sqrt{(a^2+x^2)}+\frac{2\text{A}^2a^2}{(\text{B}+2\text{A})^2}$$

$$\times\int\frac{x\dot{x}}{(b^2+x^2)\sqrt{(a^2+x^2)}}.$$ Now, by putting z^2 for a^2+x^2, the member under the integral sign is transformed to $\int\frac{\dot{z}}{z^2+b^2-a^2}$

$$=\int\frac{\dot{z}}{z^2-\frac{\text{A}a^2}{\text{B}+2\text{A}}}=\int\frac{\dot{z}}{z^2-\frac{na^2}{2}}=-\frac{1}{2a\sqrt{\frac{n}{2}}}\,.\int\frac{2a\sqrt{\frac{n}{2}}\,.\,\dot{z}}{\frac{na^2}{2}-z^2}=$$

$$-\frac{1}{2a\sqrt{\frac{n}{2}}}\text{ . hyp. log. of }\frac{a\sqrt{\frac{n}{2}}+z}{a\sqrt{\frac{n}{2}}-z}=$$ (restoring n and x)

$$-\frac{\sqrt{(\text{B}+2\text{A})}}{2a\sqrt{\text{A}}}\text{ . hyp. log. }\frac{a\sqrt{\text{A}}+\sqrt{((\text{B}+2\text{A})\,.\,(a^2+x^2))}}{a\sqrt{\text{A}}-\sqrt{((\text{B}+2\text{A})\,.\,(a^2+x^2))}};$$ hence the whole fluent is $\frac{\text{AB}a\text{M}^\circ}{\sqrt{(\text{A}+\text{B})}\,.\,(\text{B}+2\text{A})^{\frac{3}{2}}}+\frac{\text{B}x-2\text{A}\sqrt{(a^2+x^2)}}{\text{B}+2\text{A}}$

$$+\frac{a\text{A}^{\frac{3}{2}}}{(\text{B}+2\text{A})^{\frac{3}{2}}}\text{ . hy. lo. }\frac{a\sqrt{\text{A}}+\sqrt{((\text{B}+2\text{A})\,.\,(a^2+x^2))}}{a\sqrt{\text{A}}-\sqrt{((\text{B}+2\text{A})\,.\,(a^2+x^2))}};$$ which when $x=0$, is $=-\frac{2a\text{A}}{\text{B}+2\text{A}}+\frac{a\text{A}^{\frac{3}{2}}}{(\text{B}+2\text{A})^{\frac{3}{2}}}$. hyp. log. $\frac{a\sqrt{\text{A}}+a\sqrt{(\text{B}+2\text{A})}}{a\sqrt{\text{A}}-a\sqrt{(\text{B}+2\text{A})}}$; and the correct fluent (after some reductions) becomes $\frac{\text{AB}a\text{M}^\circ}{\sqrt{(\text{A}+\text{B})}\,.\,(\text{B}+2\text{A})^{\frac{3}{2}}}+\frac{\text{B}x+2a\text{A}-2\text{A}\sqrt{(a^2+x^2)}}{\text{B}+2\text{A}}$

$$+\frac{2a(\text{A})^{\frac{3}{2}}}{(\text{B}+2\text{A})^{\frac{3}{2}}}\text{ . hyp. log. }\frac{a\sqrt{\text{A}}+\sqrt{((\text{B}+2\text{A})\,.\,(a^2+x^2))}}{\sqrt{((\text{B}+\text{A})a^2+(\text{B}+2\text{A})x^2)}}\times\frac{\sqrt{(\text{A}+\text{B})}}{\sqrt{\text{A}}+\sqrt{(\text{B}+2\text{A})}}.$$

V. The only manner of shewing that the square root of a, for instance, may be either $+\sqrt{(a)}$, or, $-\sqrt{(a)}$, has been hitherto by inference, and not by actual extraction, yet it may be obtained by the latter method, as follows:

We have $a = a - 4a + 4a$; then, by the usual way,

$$\begin{array}{rl}
 & a-4a+4a \;\Big|\; \sqrt{(a)}-2\sqrt{(a)}=-\sqrt{(a)},\text{ the neg. root.}\\
 & a \\ \cline{2-2}
2\sqrt{(a)}-2\sqrt{(a)} & \Big) -4a+4a\\
 & -4a+4a \\ \cline{2-2}
 & \quad\cdot\quad\cdot \\ \cline{2-2}
\end{array}$$

THE

MATHEMATICAL REPOSITORY.

VOL. III. PART III.

MATHEMATICAL MEMOIRS, EXTRACTED FROM WORKS OF EMINENCE.

ARTICLE I.

Continuation of M. LE GENDRE'S *Memoir on Elliptic Transcendentals.*

Approximations for Elliptic functions of the second kind.

30. The formula of these functions is

$$G = \int (A + B \sin^2 \varphi) \frac{d\varphi}{\Delta}:$$

If the same substitutions be made as in Art. 27, and there be put

$$A^{\circ} = A + \frac{B}{2}, B^{\circ} = \frac{Bc^{\circ}}{2},$$

the formula will be transformed to

$$G = \frac{1 + c^{\circ}}{2} (-\frac{B}{2} \sin \varphi + G^{\circ});$$

G° being put instead of $\int (A^{\circ} + B^{\circ} \sin^2 \varphi^{\circ}) \frac{d\varphi^{\circ}}{\Delta^{\circ}}$. By new transformations we have similarly

$$G^{\circ} = \frac{1 + c^{\infty}}{2} (-\frac{B^{\circ}}{2} \sin \varphi^{\infty} + G^{\infty})$$

$$G^{\infty} = \frac{1 + c^{\circ\circ\circ}}{2} (-\frac{B^{\infty}}{2} \sin \varphi^{\circ\circ\circ} + G^{\circ\circ\circ})$$

&c.

Thus the integral formula G is easily reduced to any one we please of the successive transformed expressions G°, G^{∞}, &c.— Let us suppose that this series is continued to a very remote term G^{μ}, then we shall have $c^{\mu} = 0$, $\Delta^{\mu} = 1$, and the value of G^{μ}

will become simply $\int (A^{\mu} + B^{\mu} \sin^2 \varphi^{\mu}) d\varphi^{\mu}$. Now it is to be observed that we have $B^{\circ} = \frac{Bc^{\circ}}{2}$, and similarly $B^{\circ\circ} = \frac{B^{\circ} c^{\circ\circ}}{2}$, $B^{\circ\circ\circ} = \frac{B^{\circ\circ} c^{\circ\circ\circ}}{2}$, therefore the series B°, $B^{\circ\circ}$, &c. decreases more rapidly than the series c°, $c^{\circ\circ}$, &c. and thus in the end we may safely make $B^{\mu} = 0$. With respect to A^{μ}, since $A^{\circ} = A + \frac{1}{2}B$, $A^{\circ\circ} = A^{\circ} + \frac{1}{2}B^{\circ}$, &c. it follows that

$$A^{\mu} = A + \frac{B}{2} + \frac{Bc^{\circ}}{4} + \frac{Bc^{\circ} c^{\circ\circ}}{8} + \frac{Bc^{\circ} c^{\circ\circ} c^{\circ\circ\circ}}{16} + \&c.$$

In short, if, as before, Φ be put for the limit of the angles φ, $\frac{\varphi^{\circ}}{2}$, $\frac{\varphi^{\circ\circ}}{4}$, &c. we shall have $\Phi = \frac{\varphi^{\mu}}{2^{\mu}}$, or $\varphi^{\mu} = 2^{\mu}\Phi$.

Therefore the value of G^{μ} will be

$$G^{\mu} = 2^{\mu}\Phi\left(A + \frac{B}{2} + \frac{Bc^{\circ}}{4} + \frac{Bc^{\circ} c^{\circ\circ}}{8} + \&c.\right)$$

Now it is easy to see that the value of G will be composed of two parts, the one $= \alpha\Phi$, by supposing $\alpha = \left(A + \frac{B}{2} + \frac{Bc^{\circ}}{4} + \frac{Bc^{\circ} c^{\circ\circ}}{8} + \&c.\right)(1 + c^{\circ})(1 + c^{\circ\circ})\,\&c.$

the other is algebraic or periodical, viz.

$$-\left(\frac{1 + c^{\circ}}{2}\right)\frac{B}{2}\sin\varphi^{\circ} - \left(\frac{1 + c^{\circ}}{2}\right)\left(\frac{1 + c^{\circ\circ}}{2}\right)\frac{Bc^{\circ}}{4}\sin\varphi^{\circ\circ} - \&c.$$

But because $1 + c^{\circ} = \frac{2\sqrt{c^{\circ}}}{c}$, $1 + c^{\circ\circ} = \frac{2\sqrt{c^{\circ\circ}}}{c^{\circ}}$, &c. this second part may be put under the form $-\frac{B}{c}\left(\frac{\sqrt{c^{\circ}}}{2}\sin\varphi^{\circ} + \frac{\sqrt{c^{\circ} c^{\circ\circ}}}{4}\sin\varphi^{\circ\circ}, \&c.\right.$ Therefore the complete value of G will be

$$G = \alpha\Phi - \frac{B}{c}\left(\frac{\sqrt{c^{\circ}}}{2}\sin\varphi^{\circ} + \frac{\sqrt{c^{\circ} c^{\circ\circ}}}{4}\sin\varphi^{\circ\circ} + \frac{\sqrt{c^{\circ} c^{\circ\circ} c^{\circ\circ\circ}}}{8}\sin\varphi^{\circ\circ\circ} + \&c.\right)$$

and when $\varphi = 90^{\circ}$, then we shall have simply $G1 = \alpha\frac{\pi}{2}$. In the applications a few terms of these series will be sufficient, and

the

the terms omitted will shew the degree to which the approximation is carried.

31. These formulas apply directly to the rectification of the ellipse. Suppose always the greater axis to be unity, the excentricity $= c$; Let us consider the arc E, the origin of which is the lesser axis, and the extremity of which is determined by the ordinate to the greater axis $b \sin \varphi$, and the abcissa $\cos \varphi$, this arc is equal to the integral $\int d\varphi \sqrt{(1 - c^2 \sin^2 \varphi)}$; thus, making $A = 1$, and $B = -c^2$, we have $G = E$, hence it follows that the elliptic quadrant will be expressed by this very convergent series

$$E\,1 = \frac{\pi}{2}\left(1 - \frac{c^2}{2} - \frac{c^2 c^{\circ}}{4} - \frac{c^2 c^{\circ} c^{\infty}}{8} - \&c.\right)(1 + c^{\circ})(1 + c^{\circ\circ}), \&c.$$

The rectification of the hyperbola is comprehended in the same formulas. Let c be the semi-transverse axis, b its semi-conjugate, 1 the excentricity. Let $b^2 \tan^2 \varphi$ be an ordinate to the transverse axis, and Υ the hyperbolic arc comprehended between the vertex of the curve and this ordinate. We have $\Upsilon = \int \frac{b^2 d\varphi}{\Delta \cos^2 \varphi} =$ $\Delta \tan \varphi - \int \frac{c^2 d\varphi \cos^2 \varphi}{\Delta}$. This last integral compared with the formula G gives $A = c^2$, and $B = -c^2$, and thus we have

$$\Upsilon = \Delta \tan \varphi - G.$$

Hence it follows, that the difference between the infinite arc of an hyperbola and its asymptote is equal to G1, and has for its value

$$\frac{\pi}{4} c^2\left(1 - \frac{c^{\circ}}{2} - \frac{c^{\circ} c^{\infty}}{4} - \frac{c^{\circ} c^{\infty} c^{\circ\infty}}{8}, \&c.\right)(1 + c^{\circ})(1 + c^{\infty})(1 + c^{\circ\infty})\ \&c.$$

It does not appear that any thing more simple can be found for the rectification of the conic sections.

32. Let us now consider for a moment the relations which subsist between these different transcendentals, namely $F = \int \frac{d\varphi}{\Delta}$, $E = \int \Delta\, d\varphi$, and in general $G = \int (A + B \sin^2 \varphi) \frac{d\varphi}{\Delta}$. In the first place, since $\sin^2 \varphi = \frac{1 - \Delta^2}{c^2}$, it is evident that $G =$ $\left(A + \frac{B}{c^2}\right) F - \frac{B}{c^2} E$, and thus G is determined by means of F

and

and E. But in like manner we have $G^{\circ} = (A^{\circ} + \frac{B^{\circ}}{c^{\circ 2}})F^{\circ} - \frac{B^{\circ}}{c^{\circ 2}}E^{\circ}$; these values being substituted in the equation between G and G° (Art. 30.), at the same time taking care to substitute for A° and B° their values as expressed by A and B, there will result a new equation, which must hold true whatever be the values of A and B. This equation will give two others, the one $F = \frac{1+c^{\circ}}{2}F^{\circ}$, which we already know, and the other

$$bF = -E + \frac{1+b}{2}E^{\circ} + \frac{1-b}{2}\sin\varphi^{\circ}.$$

From this it appears that the function F of the first kind may be expressed by the two elliptic arcs E, E°. The same is therefore true of the function G in general, and of the hyperbolic arc Y in particular. In like manner we have

$$b^{\circ}F^{\circ} = -E^{\circ} + \frac{1+b^{\circ}}{2}E^{\circ\circ} + \frac{1-b^{\circ}}{2}\sin\varphi^{\circ\circ}.$$

From these two equations we may eliminate F and F°, since we have also $F = \frac{1+c^{\circ}}{2}F^{\circ}$, and there will result this relation between the three arcs E, E°, E^{∞} taken upon three consecutive ellipses if of one and the same series,

$$(1+c^{\circ})E = c^{\circ}\sin\varphi^{\circ} - \frac{b^{\circ}(1-b^{\circ})}{4}\sin\varphi^{\circ\circ} + (1+\frac{b^{\circ}}{2})E^{\circ} - \frac{b^{\circ}(1+b^{\circ})}{4}E^{\circ\circ}.$$

In the case when $\varphi = 90^{\circ}$, then $\varphi^{\circ} = 180^{\circ}$, $\varphi^{\circ\circ} = 360^{\circ}$, and the formula becomes

$$(1+c^{\circ})E1 = (2+b^{\circ})E1^{\circ} - b^{\circ}(1+b^{\circ})E1^{\circ\circ};$$

Hence it follows that the elliptic quadrant E1 may be determined by two other elliptic quadrants E1°, $E1^{\infty}$, the excentricities of which are less; and in general it appears from the formulas that in the series of ellipses having their excentricities c, c°, $c^{\circ\circ}$, &c. the rectification of any one of the ellipses may always be reduced to the rectification of two others taken at pleasure. These consequences may be found developed at greater length in the Memoirs of the Academy for 1786; but the method of approximation which thence follows is not so simple as that explained in Art. 30.

Particular developement of the formula.

$$Z = \int \frac{(f+gx^2)\,dx}{\sqrt{(\alpha^2 + 2\alpha\beta x^2\cos\theta + \beta^2 x^4)}}.$$

33. This formula frequently occurs in applications, and besides it is necessary to examine the case of imaginary factors of which we have spoken (Art. 7.)

The variable quantity x may have all values from o to infinity; but as in making $x = \infty$ we have $Z = \int \frac{g}{\beta}\, dx = \frac{g}{\beta} x$, to get rid of the term which may become infinite, we shall consider simply the formula

$$X = \int \left\{ \frac{(f + gx)\, dx}{\sqrt{(\alpha^2 + 2\alpha\beta x^2 \cos\theta + \beta^2 x^4)}} - \frac{g}{\beta}\, dx \right\},$$

which by this means will always have a finite value.

The first thing to be done is to transform this expression so that the binomial factors of the quantity under the radical may become real. For this purpose different suppositions may be made which will all equally succeed. For example

$$x = \frac{1 - y}{1 + y} \sqrt{\frac{\alpha}{\beta}},$$

$$\sqrt{(\alpha^2 + 2\alpha\beta x^2 \cos\theta + \beta^2 x^4)} = 2xy\sqrt{\alpha\beta}$$

$$\alpha + \beta x^2 = 2xy\sqrt{\alpha\beta},$$

$$\beta x^2 + \alpha \cos\theta + \sqrt{(\alpha^2 + 2\alpha\beta x^2 \cos\theta + \beta^2 x^4)} = 2\alpha y^2.$$

We shall consider only the result of the fourth supposition; it gives $x^2 = \frac{\alpha}{\beta}(y^2 - \cos\theta - \frac{\sin^2\theta}{4y^2})$, and

$$X = \int \frac{\frac{dy}{\sqrt{\alpha\beta}}\left(f - \frac{g\alpha}{\beta}\cos\theta - \frac{g\alpha \sin^2\theta}{2\beta y^2}\right)}{\sqrt{\left\{(y^2 - \cos^2 \tfrac{1}{2}\theta)(y^2 + \sin^2 \tfrac{1}{2}\theta)\right\}}},$$

where it appears that the two factors of the quantity under the radical sign are real. It also appears that the least value of y being $\cos \frac{1}{2}\theta$, we may make $y = \frac{\cos \frac{1}{2}\theta}{\cos\varphi}$, and it will follow, putting $\sin \frac{1}{2}\theta = c$, that

$$X = \int \frac{f\beta - g\alpha + 2g\alpha c^2 \sin^2\varphi}{\beta\sqrt{\beta\alpha}} \cdot \frac{d\varphi}{\sqrt{(1 - c^2 \sin^2\varphi)}}.$$

By this transformation the formula X becomes an elliptic function of the second kind to which the formlas of Art. 30 may be immediately applied, making

$$A = \frac{f\beta - g\alpha}{\beta\sqrt{\alpha\beta}},\ B = \frac{2g\alpha c^2}{\beta\sqrt{\alpha\beta}}.$$

It is to be remarked that we have supposed $\sqrt{\alpha\beta}$ real, for α and β may always be taken positive, and if the term $2\alpha\beta x^2 \cos\theta$ ought to be negative the sign — may be made to fall upon $\cos\theta$.

The immediate relation between φ and x is given by the equation

$2\alpha \sin^2 \frac{1}{2}\theta \cos^2 \varphi = -\beta x^2 - \alpha \cos \theta + \sqrt{(\alpha^2 + 2\alpha\beta x^2 \cos \theta + \beta^2 x^4)}$, from which it appears that when $x = 0$, $\varphi = 0$, and when $x = \infty$, $\varphi = 90°$. The transformed equation containing φ therefore expresses the integral X taken from $x = 0$ to the value of x which corresponds to φ,

The integral X may be expressed by means of the two simple formulas $F = \int \frac{d\varphi}{\Delta}$, $E = \int \Delta \, dp$, and in this way we have

$$X = \frac{f\beta + g\alpha}{\beta\sqrt{\alpha\beta}} F - \frac{2g\alpha}{\beta\sqrt{\alpha\beta}} E.$$

Here are some applications of the last formula, which lead to results sufficiently remarkable.

EXAMPLE 1.

34. Let it be proposed to value the two integrals $M = \int \frac{z^{-\frac{2}{3}} dz}{\sqrt{(1 - z^2)}}$, $N = \int \frac{z^{\frac{1}{3}} dz}{\sqrt{(1 - z^2)}}$, both taken from $z = 0$ to $z = 1$. It is known that the product of the two integrals $= \frac{1}{2}\pi$, (see Mem. of the Acad. 1786, page 678.)

If in the first we make $z = (1 + x^2)^{-\frac{3}{2}}$, we shall have the transformed equation $M = \int \frac{3dx}{\sqrt{(3 + 3x^2 + x^4)}}$, which it is necessary to integrate from $x = 0$ to $x = \infty$. The comparison with the formula X gives $f = 3$, $g = 0$, $\alpha = \sqrt{3}$, $\beta = 1$, $\cos \theta = \frac{1}{2}\sqrt{3}$, $c = \sin \frac{1}{2}\theta = \frac{1}{2}\sqrt{(2 - \sqrt{3})}$; whence it follows that $M = \frac{3}{\sqrt[4]{3}} F$, and as the integral ought to be taken from $x = 0$ to $x = \infty$, or from $\varphi = 0$ to $\varphi = 90°$, F will become F1 and the whole integral will be $M = \frac{3}{\sqrt[4]{3}} F1$.

In the second formula we make $z = (1 - x^2)^{\frac{1}{2}}$ and the transformed equation will be $N = \int \frac{3dx\,(1 - x^2)}{\sqrt{(3 - 3x^2 + x^4)}}$, which ought to be integrated from $x = 0$ to $x = 1$. This formula being compared with the formula X we have $f = 3$, $g = -3$, $\alpha = \sqrt{3}$, $\beta = 1$, $\cos \theta = -\frac{1}{2}\sqrt{3}$, $c = \sin \frac{1}{2}\theta = \frac{1}{2}\sqrt{(2 + \sqrt{3})}$ and hence

N =

$$N = -3x + \frac{3 - 3\sqrt{3}}{\sqrt[4]{3}} F(\varphi) + \frac{6\sqrt{3}}{\sqrt[4]{3}} E(\varphi);$$

but the relation between φ and x being

$$\frac{3 + 2\sqrt{3}}{2} \cos^2\varphi = -x^2 + \frac{3}{2} + \sqrt{(3 - 3x^2 + x_4)},$$

if we make $x = 1$, we shall have $\cos^2\varphi = 2\sqrt{3} - 3$, that is, $\tan\varphi = \sqrt{\frac{2}{\sqrt{3}}}$. But we have seen (art. 28.) that in the case of $c = \frac{1}{2}\sqrt{(2 + \sqrt{3})}$, and $\tan\varphi = \sqrt{\frac{2}{\sqrt{3}}}$, we have exactly $F(\varphi) = \frac{1}{3}F1$. Hence it is easy to conclude by the formulas of Art. 22, that $E(\varphi) = \frac{1}{3}E1 + \frac{1}{2\sqrt[4]{3}}$, therefore the integral N taken from $x = 0$ to $x = 1$ is reduced to this value

$$N = \frac{1 - \sqrt{3}}{\sqrt[4]{3}} F\,1 + \frac{2\sqrt{3}}{\sqrt[4]{3}} E\,1.$$

It is to be remarked that these values of F 1 and E 1 correspond to a modulus $c = \frac{1}{2}\sqrt{(2 + \sqrt{3})}$, while that the value of F 1 which enters into M answers to a modulus $c = \frac{1}{2}\sqrt{(2 - \sqrt{3})}$. To distinguish this last modulus, we shall design it by k, and the corresponding value of F 1 by F 1 k: thus by taking the product of the two integrals we shall have the equation

$$\frac{3}{2}\pi = \frac{3}{\sqrt[4]{3}} F\,1\,k \left\{ \frac{1 - \sqrt{3}}{\sqrt[4]{3}} F\,1 + \frac{2\sqrt{3}}{\sqrt[4]{8}} E\,1 \right\},$$

hence it follows that in this particular case the elliptic arc E1 which is a function of the second kind, may be expressed by F 1 and F 1 k, functions of the first kind. These calculations are done differently in the Memoir quoted, and we have not there deduced any result from them, not having observed that the arc E may be determined by the third of E 1. We shall find something similar in the following example.

EXAMPLE 2.

35. It is proposed to value the two integrals $P = \int \frac{z^{-\frac{1}{3}} dz}{\sqrt{(1 - z^2)}}$, $Q = \int \frac{z^{\frac{2}{3}} dz}{\sqrt{(1 - z^2)}}$ from $z = 0$ to $z = 1$: their product is known to be $\frac{3}{4}\pi$.

If in the first we make $z = (1 - x^2)^{\frac{3}{2}}$, we shall have the transformed

formed equation $P = \int \frac{3dx}{\sqrt{(3 - 3x^2 + x^4)}}$, which is to be integrated from $x = 0$ to $x = 1$. This being compared with the formula X, we have $f = 3$, $g = 0$, $\alpha = \sqrt{3}$, $\beta = 1$, $\cos\theta = -\frac{1}{2}\sqrt{3}$, $c = \frac{1}{2}\sqrt{(2 + \sqrt{3})}$, which gives $P = \frac{3}{\sqrt[4]{3}} F(\varphi)$. With respect to φ, it may be deduced from the equation $\frac{3 + 2\sqrt{3}}{2} \times \cos^2\varphi = -x^2 + \frac{3}{2} + \sqrt{(3 - 3x^2 + x^4)}$, in which, if we make $x = 1$, we shall have $\cos^2\varphi = 2\sqrt{3} - 3$, or $\tan\varphi = \sqrt{\frac{2}{\sqrt{3}}}$. This case having already occurred, we in like manner conclude that $F(\varphi) = \frac{1}{3} F 1$ and consequently $P = \frac{1}{\sqrt[4]{3}} F1$.

Let us now consider the formula Q, and in the first place, making $z^2 = y^{-3}$ we have the transformed equation $Q = \frac{3}{2}\int \frac{dy}{yy\sqrt{(y^3 - 1)}}$, to be integrated from $y = 1$ to $y = \infty$. By partial integration we have

$$Q = \frac{3}{2} \cdot \frac{\sqrt{(y^3 - 1)}}{y} - \frac{3}{4}\int \frac{ydy}{\sqrt{(y^3 - 1)}}.$$

Now let $y = 1 + x^2$, and we have $\frac{3}{4}\int \frac{ydy}{\sqrt{(y^3 - 1)}} = \frac{3}{2}\int \frac{(1 + x^2)\,dx}{\sqrt{(3 + 3x^2 + x^4)}}$, a formula which must be integrated from $x = 0$ to $x = \infty$. This being premised, the value of Q may be put under this form

$$Q = \frac{3}{2} \cdot \frac{\sqrt{(y^3 - 1)}}{y} - \frac{3}{2} x$$
$$- \frac{3}{2}\int \left\{ \frac{(1 + x^2)\,dx}{\sqrt{(3 + 3x^2 + x^4)}} - dx \right\};$$

but the part without the sign $\int$ vanishes at the beginning and at the end of the integral, thus we have simply

$$Q = -\frac{3}{2}\int \left\{ \frac{(1 + x^2)\,dx}{\sqrt{(3 + 3x^2 + x^4)}} - dx \right\}.$$

Comparing now with the formula X we have $f = -\frac{3}{2}$, $g = -\frac{3}{2}$, $\alpha = \sqrt{3}$, $\beta = 1$, $\cos\theta = \frac{1}{2}\sqrt{3}$, $c = \frac{1}{2}\sqrt{(2 - \sqrt{3})}$, and as the integral

integral ought to be taken from $\varphi = 0$ to $\varphi = 90°$, the result is (distinguishing the second c by k)

$$Q = -\left(\frac{3+3\sqrt{3}}{2\sqrt[4]{3}}\right) F1k + \frac{3\sqrt{3}}{\sqrt[4]{3}}\, E1k.$$

Now since we know that $PQ = \frac{1}{4}\pi$, this equality and that which was similarly obtained in the preceding example, give these two results

$$\frac{\pi}{4} = F1\,\left(E1k - \left(\frac{3+\sqrt{3}}{6}\right) E1k\right).$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (b')$$

$$\frac{\pi}{4} = F1k\,\left(E1 - \left(\frac{3-\sqrt{3}}{6}\right) F1\right).$$

Therefore in this case the elliptic quadrants $E1$, $E1k$, which are functions of the second kind, may be expressed by $F1$, $F1k$, functions of the first kind and consequently the same thing must hold for all the ellipses of which the excentricities are comprehended in either of the two series c, c°, $c^{\circ\circ}$, &c. k, k°, $k^{\circ\circ}$, &c. which series may also be continued to infinity in the other direction. To these relations we may join that of $F1 = \sqrt{3}F1k$, which is to be determined in the next example.

EXAMPLE III.

36. Let it be proposed to integrate the formula $R = \int \frac{dz}{(1-z^3)^{\frac{2}{3}}}$ from $z = 0$ to $z = 1$.

In the first place we make $1 - z^3 = \left(\frac{z}{y}\right)^{\frac{3}{2}}$, which gives $R = \int \frac{dy}{\sqrt{(4y^3+1)}}$, to be integrated from $y = 0$ to $y = \infty$. Next let $m = \sqrt[3]{4}$, and $my = x^2 - 1$, and we have the transformed equation $R = \frac{2}{m}\int \frac{dx}{\sqrt{(x^4-3x^2+3)}}$, which is to be integrated from $x = 1$ to $x = \infty$. This integral is easily deduced from the value of P in the preceeding example, and thence we get $R = \frac{4}{8m\sqrt[4]{3}} F1$, the modulus of $F1$ being always $c = \frac{1}{2}\sqrt{(2+\sqrt{3})}$. But there is another way of finding the value of R.

Let $1 - z^3 = \left(1 - \frac{x}{y}\right)^3$, and we immediately get the transformed equation $R = \int \frac{dy\sqrt{3}}{\sqrt{(4y^3-1)}}$, which must be integrated from $y = 1$ to $y = \infty$. Again, let $m^3 = 4$ and $my = 1 + x^2$, and we have $R = \frac{2\sqrt{3}}{m} \int \frac{dx}{\sqrt{(x^4 + 3x^2 + 3)}}$, a new formula which must be integrated from $x = \sqrt{(m-1)}$, to $x = \infty$: but by comparing with the formula X, we have $f = 1$, $g = 0$, $\alpha = 1$, $\beta = \sqrt{3}$, $\cos\theta = \frac{1}{2}\sqrt{3}$, and $c = \frac{1}{2}\sqrt{(2 - \sqrt{3})}$, besides, when $x^2 = 2m - 1$, we have $\cos^2\varphi = \frac{-m - \frac{1}{2} + \sqrt{(m^2 + m + 1)}}{\sqrt{3} - \frac{3}{2}} = \frac{(m-1)^2}{2\sqrt{3} - 3}$. But it will be found by trisection (Art. 18.) that this value of φ answers to a function $F = \frac{1}{3}F1$, whence it follows that the function F, taken from this value of φ to that of 90°, where $x = \infty$, will be $F - \frac{1}{3}F1$. Moreover, this F1, ought to be denoted by $F1k$, because the modulus is here $k = \frac{1}{2}\sqrt{(2 - \sqrt{3})}$; therefore we have $R = \frac{2\sqrt{3}}{m} \cdot \frac{1}{\sqrt{3}} \cdot \frac{2}{3} F1k$. Comparing this value with that found by the other method we shall have this very simple relation

$$F1 = \sqrt{3} \cdot F1k,$$

which joined to the two equations (b'), makes it appear that if one only of the four quantities $F1$, $F1k$, $E1$, $E1k$ be known, the three others may be determined. We have for example

$$\frac{\pi\sqrt{3}}{4} = F1 \left\{ E1 - \left(\frac{3 - \sqrt{3}}{6}\right) F1 \right\}$$

$$\frac{\pi}{4\sqrt{3}} = F1k \left\{ E1k - \left(\frac{3 + \sqrt{3}}{6}\right) F1k \right\}.$$

It appears also that a single known ellipse in the two series formed from the excentricities c and k is sufficient to determine all the others.

We particularly notice these comparisons, because there are very few cases which admit of them, and in which E can be deduced from F; here there is a double relation between the quantities E and F of one series, and those of another, formed according to a different modulus, which is yet more rare and remarkable.

Besides, as the formula R may be integrated indefinitely by both methods, it appears that the functions F, to the modulus $c = \frac{1}{2}\sqrt{(2 + \sqrt{3})}$, may be always deduced from a like function to the modulus $k = \frac{1}{2}\sqrt{(2 - \sqrt{3})}$, by taking proper amplitudes, and the same is true of ellipses of which c and k are the excentricities. These ellipses enter into the integration of the formula $\int \frac{x\,dx}{(1 - x^3)^{\frac{1}{3}}}$, which may be effected by the same two methods as those of the formula R; and thus we know two infinite series of ellipses, such, that the indefinite rectification of any one of these ellipses gives that of all the others. These reductions depend upon the peculiar nature of the radical $\sqrt{(x^4-3x^2+3)}$, which admits of being transformed to $\sqrt{(x^4+3x^2+3)}$, but it does not appear that such reductions are always practicable.

Theorem relating to definite functions of which the moduli *are the complements of each other.*

37. In the comparisons which have been established, the *moduli* c and k are such that $k=1-c^2$, or $k=b$, and thus they are the complements of each other.

If b be put instead of k, the better to represent the relation that the functions $\mathrm{E}1b$, $\mathrm{F}1b$ have to the functions $\mathrm{E}1$, $\mathrm{F}1$, which may also be denoted by $\mathrm{E}1c$, $\mathrm{F}1c$, the equations (b'), being added, give

$$\frac{\pi}{2}=\mathrm{F}1c\mathrm{E}1b+\mathrm{F}1b\mathrm{E}1c-\mathrm{F}1b\mathrm{F}1c \ldots\ldots\ldots\ldots (c').$$

This equation is demonstrated in the case of $c=\frac{1}{2}\sqrt{(2+\sqrt{3})}$; also in the case of $c=\sqrt{\frac{1}{2}}$, for then $b=c$, and it becomes

$$\frac{\pi}{2}=\mathrm{F}1\,(2\mathrm{E}1-\mathrm{F}1):$$

This has however been expressed in a different manner in the Memoirs of 1786. But these cases are only particular, and we shall now prove that the equation (c') is generally true, whatever be the value of c, which is a new and remarkable theorem relating to definite integrals.

To abridge the notation, we shall take away the 1 which indicates that the functions have an amplitude of 90°, and only distinguish the functions of b from those of c by an accent placed over the former. Then the equation to be demonstrated will be

$$\frac{\pi}{2}=\mathrm{FE}'+\mathrm{F}'\mathrm{E}-\mathrm{FF}' \ldots\ldots\ldots\ldots\ldots (d')$$

I regard the first member as unknown, and call it Π, I now difference the two members with respect to c, which is the only variable quantity they contain.

But having $\mathrm{E}=\int\Delta d\varphi$, $\mathrm{F}=\int\frac{d\varphi}{\Delta}$, $\Delta^2=1-c^2\sin^2\varphi$, we get, by known principles, $\frac{d\mathrm{E}}{dc}=-\int\frac{cd\varphi\sin^2\varphi}{\Delta}=\frac{1}{c}(\mathrm{E}-\mathrm{F})$, and $\frac{d\mathrm{F}}{dc}=\int\frac{cd\varphi\sin^2\varphi}{\Delta^3}=\frac{1}{c}\int\frac{d\varphi}{\Delta^3}-\frac{1}{c}\int\frac{d\varphi}{\Delta}$. But by the formulas of Art. 9, we have $\int\frac{d\varphi}{\Delta^3}=\frac{1}{b^2}\int\Delta d\varphi-\frac{c^2\sin\varphi\cos\varphi}{b^2\Delta}$, and in the case of $\varphi = 90^\circ$, which is that under consideration, the second term vanishes, and thus we have $\frac{dF}{dc} = \frac{E - b^2 F}{b^2 c}$. The variations of E′ and F′ are similarly expressed in respect of b, and because $bdb + cdc = 0$, we therefore get $\frac{dE'}{dc} = -\frac{1}{b}(E' - F')$, and $\frac{dF'}{dc} = \frac{-E' + c^2 F}{b^2 c}$. This premised, if we difference the equation (d'), of which the first member is supposed Π, and substitute the values which have been found for dE, dF, &c. we shall find

$$d\Pi = 0.$$

and thus the quantity Π is constant; it is therefore only necessary to find its value in one particular case.

Let $c = 0$, or $b = 1$, we then have $E = F = \frac{1}{2}\pi$, $E' = 1$; as to F′ it is infinite; for having $F' = \int \frac{d\varphi}{\cos\varphi} = \log.\tan(45^\circ + \frac{1}{2}\varphi)$, if we make $\varphi = 90^\circ$, F′ becomes infinite. But as this infinity is logarithmic, and is multiplied by $E - F = 0$, the product is also equal to 0. Therefore the second member of the equation (d') becomes in this case, simply, $\frac{\pi}{2}$; therefore it is always equal to $\frac{1}{2}\pi$, and thus the equation (d'), which is expressed more accurately by (c'), is true, whatever be the value of c.

This theorem gives a relation between the functions $E1$, $F1$, which belong to the modulus c, and the like functions to the modulus b. Another relation of this kind is necessary, to be able to determine the functions of the second species $E1b$, $E1c$ by those of the first $F1b$, $F1c$, but it does not appear that it can be generally found.

We have already remarked two cases which admit of these reductions; this new theorem furnishes a third. Let us suppose $b = c^2 = \frac{1-b}{1+b}$, which gives $b = -1 + \sqrt{2}$, $c^2 = -2 + 2\sqrt{2}$; in this case $E1b$ and $F1b$ become $E^\circ 1$ and $F^\circ 1$, and the equation (c') will give

$$\frac{\pi}{2} = F1E^\circ 1 + F^\circ 1 E1 - F1F^\circ 1;$$

but by the formulas of Art. 32, we have $F1 = (1 + c^2)F^\circ 1 = \sqrt{2}F^\circ 1$, and

$$bF1 = -E1 + (1 + b)E^\circ 1.$$

Hence it follows that the elliptic quadrants $E1$, $E^\circ 1$ may be determined by the function $F1$, alone, and that

$$\frac{\pi}{2} = \mathrm{F}1 \left\{ \mathrm{E}1 \sqrt{2} + (1 - \sqrt{2})\, \mathrm{F}1 \right\}$$

$$\frac{\pi}{2} = \mathrm{F}1 \left\{ 2\mathrm{E}^{\circ}1 - \mathrm{F}1 \right\}.$$

The same will be true of all the other elliptic functions formed according to the series of excentricities, c, c°, $c^{\circ\circ}$, &c.

Remark upon the Integral Formula.

$$\mathrm{Z} = \int \frac{(\mathrm{A} + \mathrm{B}x^2 + \mathrm{C}x^4 + \&c.)\, dx}{(1 \pm x^2)\sqrt{(\alpha + \beta x^2 + \alpha x^4)}}.$$

38. This formula may always be decomposed into two parts, one disengaged from the denominator $1 \pm x^2$, and having the form

$$\int \frac{(\mathrm{A}' + \mathrm{B}'x^2 + \mathrm{C}'x^4 + \&c.)dx}{\sqrt{(\alpha + \beta x^2 + \alpha x^4)}},$$

and the other having the form

$$\int \frac{\mathrm{A}''(1 \mp x^2)dx}{(1 \pm x^2)\sqrt{(\alpha + \beta x^2 + \alpha x^4)}}.$$

The first may be reduced easily to elliptic functions of the first and second species, the second seems, at first, to depend on functions of the third species; but by a very simple substitution, it will be found absolutely integrable by circular arcs, or by logarithms. For, let $1 \pm x^2 = \frac{2x}{z}$, and this second integral becomes

$$\int \frac{dz}{\sqrt{(4\alpha \mp 2\alpha z^2 + \beta z^2)}}.$$

Therefore the formula Z may always be integrated by elliptic arcs.

Approximation for elliptic functions of the third kind.

39. Let us now consider the functions of the third species represented by the general formula.

$$\mathrm{H} = \int \frac{\mathrm{A} + \mathrm{B}\sin^2\phi}{1 + n\sin^2\phi} \cdot \frac{d\phi}{\Delta},$$

in which the coefficients A, B, n may have any values, real or imaginary. The method we have hitherto followed consists in reducing the proposed integral formula to another of the same kind, the modulus of which is smaller, and in continuing this reduction, until we come to a modulus so small as to admit of being entirely rejected, in which case the formula integrates itself. The same method applies, with equal success to functions of the third kind, and it fortunately happens that the same substitution, which serves to diminish continually the modulus c, produces the same effect on the coefficient n, and soon renders it less than any given quantity. This is the more remarkable, that it takes place alike, whether the value of n in the proposed formula be positive or negative: however, it appears at first sight that the nature of the function should change when n is negative and greater than unity, because it may then be infinite, and indeed becomes so, as often as $1 + n\sin^2\varphi = 0$.

To put the matter in a clear light, we shall prove that whether n be positive or negative, the proposed formula may always be reduced to another, in which n is less than unity, and even less than c.

In the first place, it is to be observed that the formula H may be written thus

$$H = \frac{An+B}{2n}\int\frac{d\varphi}{\Delta} + \frac{An-B}{2n}\int\frac{1-n\sin^2\varphi}{1+n\sin^2\varphi}\cdot\frac{d\varphi}{\Delta}.$$

There being no difficulty in the first part, let us consider the second, and put

$$P = \int\frac{1-n\sin^2\varphi}{1+n\sin^2\varphi}\cdot\frac{d\varphi}{\Delta}.$$

and at the same time make

$$Q = \int\frac{1-\frac{c^2}{n}\sin^2\varphi}{1+\frac{c^2}{n}\sin^2\varphi}\cdot\frac{d\varphi}{\Delta},$$

by adding these two formulas we get

$$P+Q = \frac{1-c^2\sin^4\varphi}{1+\left(n+\frac{c^2}{n}\right)\sin^2\varphi+c^2\sin^4\varphi}\cdot\frac{2d\varphi}{\Delta}.$$

Let ψ be such an angle that $\sin\psi = \frac{(1+c)\sin\varphi}{1+c\sin^2\varphi}$, and to abridge, let $m = \frac{(n-c)^2}{n(1+c)^2}$, the substitution will give

$$P+Q = \frac{2}{1+c}\cdot\frac{d\psi}{1+m\sin^2\psi},$$

and thus we have this very remakable result

$$\int\frac{1-n\sin^2\varphi}{1+n\sin^2\varphi}\cdot\frac{d\varphi}{\Delta} + \int\frac{1-\frac{c^2}{n}\sin^2\varphi}{1+\frac{c^2}{n}\sin^2\varphi}\cdot\frac{d\varphi}{\Delta} = \frac{2}{1+c}\cdot\int\frac{d\psi}{1+m\sin^2\psi}.$$

Now, of the two numbers n, $\frac{c^2}{n}$, if one is greater than c, the other is smaller, therefore, *the integral formula* H *may always be reduced to a similar function in which* n *is less than* c.

Hence, when n is negative, and greater than unity, the inconvenience, arising from the formula H becoming infinite for certain values of φ, is in a manner thrown upon the transcendental of the inferior order $\frac{d\psi}{1+m\sin^2\psi}$, and thus there only remains a regular elliptic function to be considered, the value of which is always finite.

It may be remarked, that in the particular case of $n=c$, we have, by an absolute integration,

$$\int\frac{1-c\sin^2\varphi}{1+c\sin^2\varphi}\cdot\frac{d\varphi}{\Delta}=\frac{\psi}{1+c};$$

and in like manner, in the case of $n=-c$, if we make

$$\sin\omega=\frac{(1-c)\sin\varphi}{1-c)\sin^2\varphi}.$$

we have

$$\int\frac{1+c\sin^2\varphi}{1-c\sin^2\varphi}\cdot\frac{d\varphi}{\Delta}=\frac{\omega}{1-c},$$

These two formulas are both comprised in that of Art. 37.

Lastly, if n were imaginary, and of the form $\nu(\cos\alpha+\sqrt{1}-\sin\alpha)$, it appears by the general formula that the case where ν is of any magnitude may, at length be reduced to the case of ν less than c.

40. This being premised, let us proceed to the transformation of the formula H, which we put under the form

$$H=\int\left(A+\frac{B\sin^2\varphi}{1+n\sin^2\varphi}\right)\frac{d\varphi}{\Delta},$$

and in which we may suppose n less than c.

We make, as usual, $\sin^2\varphi=\frac{1}{2}(1+c^\circ\sin^2\varphi^\circ-\Delta^\circ\cos\varphi^\circ)$, and this substitution will give

$$H=\frac{1+c^\circ}{2}\left\{H^\circ-\frac{\frac{1}{2}B}{1+n}\int\frac{d\varphi^\circ\cos\varphi^\circ}{1+n^\circ\sin^2\varphi^\circ}\right\},$$

a formula where we have similarly $H^\circ=\int\left(A^\circ+\frac{B^\circ\sin^2\varphi^\circ}{1+n^\circ\sin^2\varphi^\circ}\right)\frac{d\varphi^\circ}{\Delta^\circ}$,

and where the new coefficients are

$$n^\circ=\frac{n(n+c^2)}{(1+b)^2(1+n)},\ A^\circ=A+\frac{\frac{1}{2}B}{1+n},\ B^\circ=B.\frac{c^2+2n+n^2}{2(1+b)^2(1+n)^2},$$

The same law takes place also in the subsequent transformations, so that, making

$$n^{\circ\circ}=\frac{n^\circ(n^\circ+c^{\circ 2})}{(1+b^\circ)^2(1+n^\circ)},A^{\circ\circ}=A^\circ+\frac{\frac{1}{2}B^\circ}{1+n^\circ},B^{\circ\circ}=B^\circ.\frac{c^{\circ 2}+2n^\circ+n^{\circ 2}}{2(1+b^\circ)^2(1+n^\circ)^2}.$$

we shall have

$$H^{\circ} = \frac{1+c^{\circ\circ}}{2}\left\{H^{\circ\circ} - \frac{\frac{1}{2}B^{\circ\circ}}{1+n^{\circ}}\int\frac{d\varphi^{\circ\circ}\cos\varphi^{\circ\circ}}{1+n^{\circ}\sin^2\varphi^{\circ\circ}}\right\},$$

and so on.

41. Let us now enquire particularly, what is the law of the progression of the co-efficients n, n°, $n^{\circ\circ}$, &c. If we make $n = mc$, and similarly $n^{\circ} = m^{\circ}c^{\circ}$, we get $m^{\circ} = \frac{m(m+c)}{1+cm}$ which is a formula sufficiently simple for the determination of m° from m; and in like manner $m^{\circ\circ}$ is found from m°, &c. Now since we suppose $n < c$, we have $m < 1$. But from thence it follows that we have also $m^{\circ} < 1$, and even $m^{\circ} < m$, for, let us for a moment make $m^{\circ} = \delta m$, or $\delta = \frac{m+c}{1+cm}$, then it will follow that $1+\delta = \frac{(1+c)(1+m)}{1+cm}$, and $1-\delta = \frac{(1-c)(1-m)}{1+cm}$; these values shew that $1+\delta$ and $1-\delta$ are always positive, and, therefore, that δ is always less than unity. Therefore $m^{\circ\circ} < m^{\circ}$, $m^{\circ\circ\circ} < m^{\circ\circ}$, &c.

Hence it appears not only that the terms m, m°, $m^{\circ\circ}$, are less than unity, but that they decrease continually; from which it follows that the series n, n°, $n^{\circ\circ}$, $n^{\circ\circ\circ}$, &c. decreases more rapidly than the series c, c°, $c^{\circ\circ}$, &c. every term of the former being less than the corresponding term of the latter.

With respect to the signs of the coefficients, there are two cases to be considered:

1°. If n is positive, or that being negative it is greater than c^2, then n°, will be positive; the same will be true of $n^{\circ\circ}$, and thus all the terms n°, $n^{\circ\circ}$, $n^{\circ\circ\circ}$, &c. without exception, will be positive. The first case contains the particular instance of $n = \pm c$, which gives $n^{\circ} = c^{\circ}$, $n^{\circ\circ} = c^{\circ\circ}$, and so on to infinity; but it is needless to consider this, as it appears from Art. 38, that then the function H is reducible to the first species.

2°. If n is negative and less than c^2, let $n = -c^2\sin^2\theta$, we may, in like manner, make $n^{\circ} = -c^{\circ 2}\sin^2\theta^{\circ}$, and we shall find that the angle θ is real, and determined by the formula

$$\tan(\theta^{\circ}-\theta) = b\tan\theta,$$

here the law is the same as that by which φ° is derived from φ. Therefore, calculating θ°, θ°, $\theta^{\circ\circ}$, successively by this law, we have $n^{\circ} = -c^{\circ 2}\sin^2\theta^{\circ}$, $n^{\circ\circ} = -c^{\circ\circ 2}\sin^2\theta^{\circ\circ}$, from which it appears, that in the second case, all the quantities n, n°, &c. are negative, and smaller than the corresponding quantities c, c°, &c. which renders the series n, n°, $n^{\circ\circ}$, &c. yet more converging.

42. The law of the coefficients n being determined, we come now to that of the co-efficients A and B. In order to abridge, let

$$k = \frac{c^2+2n+n^2}{2(1+b)^2(1+n)^2},$$

and let k° denote a like function of c° and n°, and so on, we have therefore

$$B^{\circ} = kB,\ B^{\circ\circ} = kk^{\circ}B,\ B^{\circ\circ\circ} = kk^{\circ}k^{\circ\circ}B,\ \&c.$$

But since at last the numbers c and n, which may be denoted by c^{μ} and n^{μ}, are so small as to admit of being considered as $= 0$, it is evident that k^{μ} is also $= 0$, and therefore, that we may safely make $B^{\mu} = 0$.

As to the value of A^{μ}, it will readily appear to be given by the indefinite series

$$A^{\mu} = A + \frac{\frac{1}{2}B}{1+n} + \frac{\frac{1}{2}kB}{1+n^{\circ}} + \frac{\frac{1}{2}kk^{\circ}B}{1+n^{\circ\circ}} + \&c.$$

which converges rapidly, and of which more or fewer terms may be calculated according to the degree of approximation that may be wished for. Let Φ always denote the limit of the angles φ, $\frac{\varphi^{\circ}}{2}$, $\frac{\varphi^{\circ\circ}}{2}$, &c. from which it follows, that after a sufficient number of terms $\varphi^{\mu} = 2^{\mu}\Phi$; then the value of H^{μ} will be $= 2^{\mu} A^{\mu} \Phi$.

43. It is now easy to develope the complete value of H which results from the successive transformations. To abridge, let

$$\alpha = \left(A + \frac{\frac{1}{2}B}{1+n} + \frac{\frac{1}{2}kB}{1+n^{\circ}} + \frac{\frac{1}{2}kk^{\circ}B}{1+n^{\circ\circ}} + \&c.\right)(1+c^{\circ})(1+c^{\circ\circ})\ \&c.$$

and we shall have

$$\begin{aligned} H = \alpha\Phi &- \frac{1+c^{\circ}}{2} \cdot \frac{\frac{1}{2}B}{1+n}\int\frac{d\varphi^{\circ}\cos\varphi^{\circ}}{1+n^{\circ}\sin^{2}\varphi^{\circ}} \\ &- \frac{1+c^{\circ}}{2} \cdot \frac{1+c^{\circ\circ}}{2} \cdot \frac{\frac{1}{2}kB}{1+n^{\circ}}\int\frac{d\varphi^{\circ\circ}\cos\varphi^{\circ\circ}}{1+n^{\circ\circ}\sin^{2}\varphi^{\circ\circ}} \\ &- \frac{1+c^{\circ}}{2} \cdot \frac{1+c^{\circ\circ}}{2} \cdot \frac{1+c^{\circ\circ\circ}}{2} \cdot \frac{\frac{1}{2}kk^{\circ}B}{1+n^{\circ\circ}}\int\frac{d\varphi^{\circ\circ\circ}\cos\varphi^{\circ\circ\circ}}{1+n^{\circ\circ\circ}\sin^{2}\varphi^{\circ\circ\circ}} \\ &- \&c. \end{aligned}$$

The terms which remain to be integrated depend on arcs of a circle, when all the co-efficients n°, $n^{\circ\circ}$, &c. are positive, and on logarithms when they are negative. When $\varphi = 90°$, or any multiple of 90°, these integrals disappear, and then we have simply $H = \alpha\Phi$, therefore $H\ 1 = \alpha \cdot \frac{1}{2}\pi$.

If we make $n = 0$, then $k = \frac{c^{\circ}}{2}$, $k^{\circ} = \frac{c^{\circ\circ}}{2}$, &c. and the value of H agrees with that of G in art. 30.

It is to be remarked that the value of H terminates, as well as that of the limit A^{μ}, if any one of the numbers $k, k^{\circ}, k^{\circ\circ}, k^{\circ\circ\circ}$, &c. be equal to 0, or if we have one of these equalities $n = -1 + b$, $n^{\circ} = -1 + b^{\circ}$, $n^{\circ\circ} = -1 + b^{\circ\circ}$, &c. We therefore know an infinity of cases where the function H may be reduced to the first species. The cases alluded to are those where, by making $n=-c^2\sin^2\theta$, we have for θ one of the values which satisfy the equation $F(\theta)=\frac{2\nu+1}{2^\mu}F1$, μ and ν being any whole numbers whatever.

Second method of transformation applied to the function H:

44. This other method consists in taking, as in art. 39, a new angle φ' such that

$$\sin\varphi'=\frac{(1+c)\sin\varphi}{1+c\sin^2\varphi}.$$

This angle φ' increases continually with the angle φ, and they coincide entirely when φ is a multiple of 90°. Let us, as usual, make $c'=\frac{2\sqrt{c}}{1+c}$, and $\Delta'=\sqrt{(1-c'^2\sin^2\varphi')}$, then we have

$$\sin^2\varphi=\frac{1-\Delta'}{c(1+\Delta')},\quad \frac{d\varphi}{\Delta}=\frac{1}{1+c}\cdot\frac{d\varphi'}{\Delta'},$$

these values being substituted in the formula

$$H=\int\left(A+\frac{B\sin^2\varphi}{1+n\sin^2\varphi}\right)\frac{d\varphi}{\Delta},$$

give

$$H=\frac{1}{1+c}H'-\frac{B}{2n(1+c)}\int\frac{d\varphi'}{1+n'\sin^2\varphi'},$$

H′ representing $\int\left(A'+\frac{B'\sin^2\varphi'}{1+n'\sin^2\varphi'}\right)\frac{d\varphi'}{\Delta'}$, and the new coefficients being

$$A'=A+\frac{B}{2n},\quad B'=\frac{B(n^2-c^2)}{2n^2(1+c)^2},\quad n'=\frac{(c-n)^2}{n(1+c)^2}.$$

Thus we find that the formula H depends on a similar formula H′ in which c and n are different. We must now examine whether this transformation has any advantage.

With respect to c, the change appears disadvantageous, because that c' being greater than c, the radical Δ' differs more from unity than Δ. However, if the transformations be repeated, the limit of the quantities c, c', c'', &c. being unity, we may take $\cos\varphi$ for the last radical Δ, and then the integration may be accomplished by known methods. But this kind of approximation, which we have not hitherto noticed, is not the most convenient, and it is much better that the successive transformations should tend to diminish c. Now it is evident that this effect may be produced by reversing the preceding result, or, which comes to the same, by writing it thus:

$$H = (1 + c^\circ)\, H^\circ + \frac{B'}{2n^\circ}\int \frac{d\varphi}{1 + n \sin^2\varphi},$$

then we have

$$\sin\varphi = \frac{(1 + c^\circ)\sin\varphi^\circ}{1 + c^\circ \sin^2\varphi^\circ},\ H^\circ = \int\left(A^\circ + \frac{B^\circ \sin^2\varphi^\circ}{1 + n^\circ \sin^2\varphi^\circ}\right)\frac{d\varphi^\circ}{\Delta^\circ}.$$

and the new co-efficients are

$$n^\circ = \left(\frac{1 + c^\circ}{2}\right)^2 \left\{2n + c^2 \pm 2\sqrt{(n^2 + nc^2)}\right\},$$

$$B^\circ = \pm B \cdot \frac{2n^\circ}{\sqrt{(n^2 + nc^2)}},$$

$$A^\circ = A \mp \frac{B}{\sqrt{(n^2 + nc^2)}}.$$

Because of the double sign ± we must choose between two transformations, and it seems convenient to take that where n° is the smallest. But the values of n° are such that their product $= c^{\circ 2}$. Thus we may always so manage, that in the transformed expressions, n° shall be smaller than c°, however great n may be. The same advantage may also be obtained in the ulterior transformations; so that by this method we may readily find a value of H as near to its true value as we please, without any preparation being necessary when n is somewhat great. However this method is in some respects inferior to the first, inasmuch as the value of n° is irrational and that it even becomes imaginary if n is negative and $< c^2$. A second disadvantage of this method is, that φ° cannot be deduced conveniently from φ but by employing an auxiliary angle ψ, such, that $\psi = c \sin\varphi$; then we have $\sin\varphi^\circ = \frac{1}{\sqrt{c^\circ}} \tan\frac{1}{2}\psi$.

Third method of transformation applied to the formula H.

45. The method we have followed in art. 20, will apply to the formula H. For this purpose it must be observed that if, according to art. 17, we make

$$\sin^2\varphi = \frac{1 - \cos\psi}{1 + \sqrt{(1 - c^2 \sin^2\psi)}},$$

we shall have $\frac{d\varphi}{\Delta(\varphi)} = \frac{\frac{1}{2}d\psi}{\Delta(\psi)}$: substituting these values in the formula of art. 40, and in order to abridge, putting

$$k = \frac{c^2 - n^2}{n^2 + 2n + c^2}, \text{ we shall find}$$

$$H = \int\left\{A + \frac{B(1 + n + nk - nk\sin^2\psi)}{(n^2 + 2n + c^2)(1 - k^2 + k^2\sin^2\psi)}\right\} \cdot \frac{\frac{1}{2}d\psi}{\Delta(\psi)}$$
$$- \frac{\frac{1}{2}B}{n^2 + 2n + c^2}\int\left(\frac{(n + k + kn)\,d\psi\cos\psi}{(1 - k^2 + k^2\sin^2\psi)\Delta(\psi)} + \frac{d\psi}{1 + k\cos\psi}\right).$$

The last part is integrable by arcs of a circle, and by logarithms. The first is a formula, similar to H, in which c is the same, and n different. The new value of n being called ν, we shall have

$$\nu = \frac{k^2}{1-k^2} = \frac{(c^2-n^2)^2}{4n(n+1)(n+c^2)}.$$

It would be necessary to reverse the preceding result, if we wished to deduce from it an approximation similar to that indicated in art. 20; but as this method is much less convenient than the two preceding, we shall not insist on it farther, and, in concluding, only observe that if the integral H is known when $n = \alpha$, it will also be known when

$$n = \frac{(c^2-\alpha^2)^2}{(\alpha^2+2\alpha+c^2)^2-(c^2-\alpha^2)^2}.$$

And by the reverse of this formula we shall find that if the integral H is known when $n = \beta$, it will also be known in the four cases contained in the expression

$$n = \frac{-\sqrt{\beta} \pm \sqrt{(\beta+c^2)}}{\sqrt{\beta} \pm \sqrt{(\beta+1)}}.$$

This also appears, by combining the two first methods, for the first gives this relation between φ and φ°, $\frac{d\varphi}{\Delta(\varphi)} = \frac{1+c^\circ}{2} \cdot \frac{d\varphi^\circ}{\Delta^\circ}$, or $F(c, \varphi) = \frac{1+c^\circ}{2} \cdot F(c^\circ, \varphi^\circ)$; the second, by putting ψ in place of φ, gives $F(c, \psi) = (1+c^\circ) F(c^\circ, \psi^\circ)$; therefore, combining the two, we have $F(c, \varphi) = \frac{1}{2} F(c, \psi)$, or $\frac{d\varphi}{\Delta(\varphi)} = \frac{\frac{1}{2}d\psi}{\Delta(\psi)}$; this is the foundation of the third method, and therefore the conclusions drawn from this method are in effect contained in the two others.

We now propose to shew, by the help of these transformations, in what cases the elliptic function H may be reduced to functions of the second kind, so as to depend on elliptic arcs only.

Of the cases in which the function H *may be reduced to elliptic functions of the second kind.*

46. The most simple cases, where the reduction has place immediately, are those of $n = -1$, $n = -c^2$, $n = \pm c$ (see art. 10 and 39). If, therefore among the successive transformed expressions obtained by one of the three methods we have explained, one is found which agrees with one of these cases, we may be certain that the proposed formula, and all the

others resulting from the transformations, the number of which is infinite, may be reduced to the second species, sometimes even to the first, and thus will depend on elliptic arcs only. Hence it appears that it is possible to find an infinity of values of n, real and imaginary, expressed as functions of c, which give occasion to this reduction. This is what we are now to explain.

By the first method. From the value of n corresponding to c we draw this value of n° corresponding to c°,

$$n^\circ = \frac{n(n + c^2)}{(1 + b)(1 + n)}.$$

Reciprocally, from the value of n° corresponding to c° we deduce this value of n corresponding to c:

$$n = \frac{2}{(1 + c^\circ)^2} \left\{ n^\circ - c^\circ \pm \sqrt{(n^\circ + 1)(n^\circ + c^{\circ 2})} \right\}:$$

the use of these formulas is now to be shewn.

If in the first we make $n = -1$, or $n = -c^2$, we have no result, if we make $n = \pm c$, we have $n^\circ = \frac{1 - b}{1 + b} = c^\circ$, whence $n = c$, from which we learn nothing new.

If in the second we make $n^\circ = -1$, we have $n = \frac{-2}{1 + c^\circ}$. The second member is a function of c°, but it must be made a function of c, and to effect this, we must put for c° its value $\frac{1 - b}{1 + b}$; we thus get $n = -1 - b$, which is a new case. But as, according to art. 39, the values of n which are less than c are to be considered, and those which exceed c are to be reduced to the value $\frac{c^2}{n}$, in this case, instead of the value $-1 - b$, we may take $\frac{c^2}{-1 - b}$, or $-1 + b$. The same result is also obtained immediately by making $n^\circ = -c^\circ$ for then we get $n = -1 + b$.

The case $n = -1 + b$ may lead to an infinite number of of others. Let us make in the second formula $n^\circ = -1 + b^\circ$, and we have

$$n = \frac{-1 + b^\circ - c^\circ \pm b^\circ \sqrt{(1 - b^\circ)}}{\frac{1}{2}(1 + c^\circ)^2}.$$

The second member must be changed into a function of c or b, and to do this, we must make $b^\circ = \frac{2\sqrt{b}}{1 + b}$, $c^\circ = \frac{1 - b}{1 + b}$, which will give

$$n = \frac{-(1-\sqrt{b})}{1 \pm \sqrt{\left(\frac{b}{1+b}\right)}}.$$

From these two values of n, four others may be found by a new substitution, and from these eight others may be found, and so on.

All the values of n thus found are negative, and less than c^2, they may therefore be represented by the formula $n = -c^2 \sin^2 t$. We have already remarked, in art. 43, that the values which make some one of the quantities $n + 1 - b$, $n^{\circ} + 1 - b^{\circ}$, $n^{\circ\circ} + 1 - b^{\circ\circ}$, &c. equal to 0, at the same time also make one of those of the series k, k°, $k^{\circ\circ}$, &c. equal to 0, and thus, in all these cases, the general formula of art. 43, will terminate of itself, and the function H be reduced to the first species.

Again, in the second formula, let $n^{\circ} = c^{\circ}$, then $n = \pm c$, here we get no new result; but let $n^{\circ} = -c^{\circ}$, and we shall have for new values $n = -c^2 \pm bc\sqrt{-1}$, hence, by repeated substitutions, there results an infinity of others.

In general, it appears that every value of n which is negative and greater than c^2 gives two imaginary values, and that every positive value gives two real values, the one positive and the other negative. But among the positive values the only case we yet know is that of $n = c$, which does not lead to any more; the other methods furnish new cases.

47. *By the second method.* From a value of n corresponding to c we draw this value of n° corresponding to c°,

$$n^{\circ} = \left(\frac{\sqrt{(n + c^2)} - \sqrt{n}}{1 + -b}\right)^2.$$

Reciprocally, from a value of n° corresponding to c°, we get this value of n corresponding to c:

$$n = \frac{(c^{\circ} - n^{\circ})^2}{n^{\circ}(1 + c^{\circ})^2}.$$

In the first formula let $n = c$, and we shall have $n^{\circ} = c\left(\frac{\sqrt{(1+c)} - 1}{1+b}\right)^2$; but it is necessary to change the second member into a function of c, and for this purpose, to make $c = \frac{2\sqrt{c^{\circ}}}{1+c}$, $b = \frac{1-c^{\circ}}{1+c^{\circ}}$. The substitution being made, the index o is to be taken away, and the result will be

$$n = \tfrac{1}{2}\sqrt{c}\left\{1 + \sqrt{c} - \sqrt{(1+c)}\right\}^2.$$

If we had taken $n = -c$, we should have found this value of n,

$$n = -\tfrac{1}{2}\sqrt{c}\left\{1 - \sqrt{c} - \sqrt{(1+c)}\right\}^2.$$

These two new values are less than c, and each may furnish an infinity of others, by the formula of this second method as well as by that of the first.

48. *By the third method.* Supposing c to remain the same, if ν is one value of n, another is known by the formula

$$n = \frac{(c^2 - \nu^2)^2}{4(\nu + 1)(\nu + c^2)},$$

and also another by the inverse of this formula, which gives

$$n = \frac{\pm\sqrt{\left(1 + \frac{c^2}{\nu}\right)} - 1}{\sqrt{\left(1 + \frac{1}{\nu}\right)} + 1}.$$

In this we may put $\frac{c^2}{\nu}$ instead of ν, and there will result a a third formula

$$n = \frac{\pm\sqrt{(1 + \nu)} - 1}{\sqrt{\left(1 + \frac{\nu}{c^2}\right)} + 1}.$$

This third method is, as we have already remarked, a consequence of the two others, but it is better adapted to the object in view, because the value of n is expressed immediately as a function of c, and because there is no reduction to make from a function of c° to a function of c.

The first formula producing nothing in the more simple cases, we go to the second, and making $\nu = -1$, we get $n = \pm b - 1$, but this value is already known. Let us now make $\nu = b - 1$ in the third formula, and we shall have

$$n = \frac{\pm\sqrt{b} - 1}{\sqrt{\left(\frac{b}{b - 1}\right)} + 1},$$

a value which has been already found by the first method, but less directly. Here the sign of b may be changed, and there will result two imaginary values.

Let $\nu = -c^2$ in the second formula, $\nu = -1$ in the third, and we shall have $n = -c^2 \pm bc\sqrt{-1}$, a result which is already known.

In short, let us make $\nu = c$, in the second or third formula, and we shall have

$$n = \frac{\pm\sqrt{(1 + c)} - 1}{\sqrt{\left(1 - \frac{1}{c}\right)} - 1},$$

a value which is the same as we have found by the second method. If we had made $\nu = -c$, we should have had thes imaginary values

$$n = \frac{\pm\sqrt{(1-c)}-1}{\sqrt{\left(1-\frac{1}{c}\right)}-1}.$$

It is useless to enter into greater details, and it appears that by the combination of these formulas we may obtain an infinite number of values of n expressed by functions of c, real or imaginary, by means of which the formula H may be freed from the denominator $1 + n \sin^2\varphi$ by one or several suitable transformations, and which may be effected by the preceding methods. Thus this elliptic function of the third kind may be reduced, in an infinity of cases, to a function of the second kind, so as to depend only on arcs of an ellipse.

Another manner of valuing elliptic functions.

49. It may be necessary in certain cases, particularly in problems of mechanics, to express the elliptic functions on which the problems depend in terms of the variable quantity φ alone.

Then the developement may be made in the usual manner; but the determination of the co-efficients may always be greatly simplified by the preceding methods.

Let us consider in the first place the function $F = \int \frac{d\varphi}{\Delta}$, and let us suppose $\frac{1}{\Delta} = A - 2B\cos 2\varphi + 4C\cos 4\varphi - \&c.$ in order that we may have

$$F = A\varphi - B\sin 2\varphi + C\sin 4\varphi - D\sin 6\varphi + \&c.$$

The values of the co-efficients A, B, C, D, &c. are now to be found in as simple a manner as is possible. But, by a first developement $\frac{1}{\Delta} = 1 + \frac{1}{2}c^2\sin^2\varphi + \frac{1\cdot 3}{2\cdot 4}c^4\sin^4\varphi + \&c.$ Afterwards, putting instead of the powers of the sines their values in terms of the linear cosines we find a series of the required form of which the co-efficients are

$$\begin{array}{lllllll}
A = 1 + \frac{1}{4}c^2 + & \frac{1\cdot 9}{4\cdot 16}c^4 + & & \frac{1\cdot 9\cdot 25}{4\cdot 16\cdot 36}c^6 + & & \frac{1\cdot 9\cdot 25\cdot 49}{4\cdot 16\cdot 36\cdot 64}c^8 & + \&c. \\
B = \quad \frac{1}{2}S + & \frac{2}{3}S & + & \frac{3}{4}S & + & \frac{4}{5}S & + \&c. \\
C = & \frac{1}{8}S & + & \frac{2}{10}S & + & \frac{3}{12}S & + \&c. \\
D = & & & \frac{2}{18}S & + & \frac{4}{21}S & + \&c. \\
\&c. & & & \&c. & & &
\end{array}$$

In each value, the letter S represents the term above it of the preceding value. And by this means, the law of these series is sufficiently evident. The multiplicators to be employed in passing from one line to another are in general $\frac{n}{(n+1)(2n+2)}$, $\frac{2n}{(n+1)(2n+3)}$, $\frac{3n}{(n+1)(2n+4)}$, &c. Their law does not serve to find B from A, but it serves to find C from B by making $n = 1$, D from C by making $n = 2$, &c. We may, therefore, by these series calculate completely the values of A, B, C, &c. but when c differs little from unity, a great number of terms would be required to have a moderately accurate approximation.

50. The value of $\frac{1}{\Delta}$ being multiplied successively by $d\varphi$, $d\varphi \cos 2\varphi$, $d\varphi \cos 4\varphi$, &c. and then integrated from $\varphi = 0$ to $\varphi = 90°$, will give

$$A \cdot \frac{\pi}{2} = \int \frac{d\varphi}{\Delta}, \; B \cdot \frac{\pi}{2} = \int - \frac{d\varphi \cos 2\varphi}{\Delta}, \; 2C \cdot \frac{\pi}{2} = \int \frac{d\varphi \cos 4\varphi}{\Delta}, \text{ \&c.}$$

Thus each of the co-efficients A, B, &c. is found by a definite integral, and these integrals depend only on the quantities $F1$, $E1$. We have immediately for the two first $A\frac{\pi}{2} = F1$, and $B\frac{\pi}{2} = \frac{2}{c^2} \times (F1 - E1) - F1$, and by substituting the values of $F1$ and $E1$ (Art. 27 and 31) we find

$$A = (1 + c^{\circ})(1 + c^{\circ\circ})(1 + c^{\circ\circ\circ}), \text{ \&c.}$$

$$\frac{B}{A} = \frac{c^{\circ}}{2} + \frac{c^{\circ} c^{\circ\circ}}{4} + \frac{c^{\circ} c^{\circ\circ} c^{\circ\circ\circ}}{8} + \text{ \&c.} \quad \ldots\ldots\ldots\ldots (\alpha)$$

These two first co-efficients being found, all the rest may be deduced from them; for by comparing the differential of $\frac{1}{\Delta}$ with that of its value, we obtain

$$\begin{aligned} 2 \cdot 3\, C &= 4\, B \left(\frac{2 - c^2}{c^2}\right) - A, \\ 3 \cdot 5\, D &= 16\, C \left(\frac{2 - c^2}{c^2}\right) - 1 \cdot 3\, B \quad \ldots\ldots\ldots (\beta) \\ 4 \cdot 7\, E &= 36\, D \left(\frac{2 - c^2}{c^2}\right) - 2 \cdot 5\, C, \\ &\text{\&c.} \end{aligned}$$

However, the accuracy of the result obtained from these equations is not altogether to be relied on, if c is very small, for then A and B must have been calculated to a very great degree of accuracy, if the errors produced in the other co-efficients do not soon become considerable. In this case, the series of the preceding article may be employed; but it will still be better to make use of the equations (β) themselves in an inverse order.

For example, if the approximation does not require to go beyond the term D, we may neglect E in the third of the equations (β) and we shall have the ratio of D to C. The second will give the ratio of C to B, and the first the ratio of B to A, which may serve as a verification. It appears that thence all the co-efficients are known unto that which admits of being neglected. Besides, the co-efficient A is that which requires to be determined with the greatest precision, for $A\varphi$ is the principal part of F, and that which increases indefinitely. The others are only inequalities, or *equations* of that mean value.

51. Elliptic functions of the second species may be represented by the formula $G = \int (\alpha + \beta \cos 2\varphi) \frac{d\varphi}{\Delta}$, it is easy to see that

$$G = \alpha[A\varphi - B \sin 2\varphi + C \sin 4\varphi - D \sin 6\varphi + \&c.]$$
$$- \beta [B\varphi - \frac{A+2C}{2} \sin 2\varphi + \frac{B+3D}{4} \sin 4\varphi - \frac{2C+4E}{6} \sin 6\varphi + \&c.];$$

thus the developement does not present any new difficulty, and may be made by the same co-efficients.

With respect to functions of the third kind, it will be sufficient to consider the formula $H = \int \frac{1}{1+n \sin^2\varphi} \cdot \frac{d\varphi}{\Delta}$: but if we suppose

$$H = M\varphi - N \sin 2\varphi + P \sin 4\varphi - Q \sin 6\varphi + \&c.$$

and take the differential of every member; a value of $\frac{1}{\Delta}$ will be found, which being compared with that of article 49, will give

$$N = - M + \frac{2}{n} (A - M),$$

$$2P = -2N - M + \frac{4}{n} (B - N),$$

$$3Q = -4P - N + \frac{8}{n} (C - P),$$

$$4R = -6Q - 2P + \frac{12}{n} (D - Q),$$

&c.

Hence it follows, that by supposing A, B, C, &c. known, it is sufficient to determine the first co-efficient M, to find thence all the others. But by article 43 we have

$$\frac{M}{A} = 1 - \tfrac{1}{2}n\left[\frac{1}{1+n} + \frac{k}{1+n''} + \frac{kk^{\circ}}{1+n'''} + \&c.\right]:$$

we may also deduce M from A, B, C, &c. for we have the formula $\frac{\sqrt{(1+n)}}{1+n\sin^2\varphi} = 1 + 2\alpha\cos 2\varphi + 2\alpha^2\cos 4\varphi$ &c. in which $\alpha = \frac{\sqrt{(1+n)}-1}{\sqrt{(1+n)}+1}$; multiplying this formula by $\frac{d\varphi}{\Delta}$, and integrating from $\varphi = 0$ to $\varphi = 90°$ we have

$$M\sqrt{(1+n)} = A - 2B\alpha + 4C\alpha^2 - 6D\alpha^3 + \&c.$$

52. In order to facilitate these determinations it only remains to construct a table of the values of A and B, or of their logarithms, for different values of the modulus c, which may easily be done by the formulas (α). It may be convenient to represent each by the sine of an angle, and then the complement of the modulus answers to the complement of the angle. Let A and B be the numbers of the modulus $\sin\theta$, A′ and B′ those of its complement $\cos\theta$, then, according to the theorem of art. 37, we have this relation, which may serve for continual verification, $\frac{4}{\pi AA'} = 1 - \frac{B}{A}\sin^2\theta - \frac{B'}{A'} \times \cos^2\theta$. What follows is a slight specimen of such a table, which has been carefully verified, and should be extended farther, but which cannot fail to be useful on many occasions:

Angle of modulus.	Log. of A.	Log. of B.	Angle of modulus.	Log. of A.	Log. of B.
0*d*	*Zero.*	*Inf. neg.*	90*d*	*Inf. pos.*	*Inf. pos.*
1	0,0000331	5,5807199	89	0,5390725	0,3399469
2	0,0001323	6,1829453	88	0,4799075	0,2426870
3	0,0002978	6,5354036	87	0,4412352	0,1742911
6	0,0011920	7,1389540	84	0,3663934	0,0280475
9	0,0026860	7,4936265	81	0,3164715	9,9175495
12	0,0047846	7,7470016	78	0,2778878	9,8229717
15	0,0074955	7,9453400	75	0,2460561	9,7374504
18	0,0108284	8,1092576	72	0,2188233	9,6576703
21	0,0147959	8,2497638	69	0,1949916	9,5817130
24	0,0194130	8,3734429	66	0,1738201	9,5083232
27	0,0246981	8,4845561	63	0.1548157	9.4365965
30	0,0306733	8,5860288	60	0,1376327	9,3658278
33	0,0373648	8,6799638	57	0,1220183	9.2954244
36	0.0448034	8,7679334	54	0,1077813	9,2248545
39	0,0530260	8,8511530	51	0,0947747	9,1536124
42	0,0620766	8,9305914	48	0,0828812	9,0811882
45	0,0720074	9,0070451	45	0.0720074	9,0070451

To facilitate the construction of this table, it is proper to remark that having accurately $A = \frac{2\sqrt{c^{\circ}}}{c} A^{\circ}$, we have nearly $\frac{B}{A} = \frac{c^{\circ}}{2} \sqrt{A^{\circ}}$; and this value only errs in excess by the small part $\frac{1}{16} c^{\circ 2}$, hence, the error will not be so much as a decimal unit of the 7th order, so long as c does not exceed 52° 40′, and with the correction, we may suppose c still greater. But as the value of A is calculated by successive factors of which the first is $\frac{2\sqrt{c^{\circ}}}{c}$, the value of log. A° is known immediately by the calculation of A, and thus we have B in a very expeditious way. The same property may also be applied beyond the fixed limit; for we have equally $A = \frac{2\sqrt{c^{\circ}}}{c} \cdot \frac{2\sqrt{c^{\circ\circ}}}{c^{\circ}} A^{\circ\circ}$, and hence it follows, with a greater degree of approximation, that $\frac{B}{A} = \frac{c^{\circ}}{2} \times (1 + \frac{c^{\circ\circ}}{2} \sqrt{A^{\circ\circ}})$. This value does not differ from the truth by more than two units of the 8th order, if we have $c = 84^{\circ}$: But it is not necessary to go farther, or even so far, because, when c is nearly equal to unity, A and B may be calculated by formulas which converge faster.

One of the chief uses of these numbers is to determine the functions $F1$ and $E1$ by the formulas $F1 = \frac{\pi}{2} A$, $E1 = \frac{\pi}{2} A - \frac{\pi}{4} c^2 (A + B)$. Thus, in the case where c is not very near equal to unity, we have at once $F1 = \frac{\pi}{2} (1 + c^{\circ}) A^{\circ}$, and $E1 = \frac{\pi}{2} \cdot \frac{A^{\circ}}{1 + c^{\circ}} (1 + c^{\circ 2} - c^{\circ 2} \sqrt{A^{\circ}})$, but this object is readily obtained by employing the exact formula

$$E1 = \frac{\pi}{2} \cdot \frac{A^{\circ}}{1 + c^{\circ}} \left[1 - \frac{c^{\circ 2} c^{\circ\circ}}{2} - \frac{c^{\circ 2} c^{\circ\circ} c^{\circ\circ\circ}}{4} - \&c.\right].$$

which is the same as that of art. 31, but under another form. This formula however is not convenient in practice, if c is nearly equal to unity. In that case b will be very little and it will be proper to employ the formula

$$E\,1 = 1 + \tfrac{1}{2} b^2 (\log. \tfrac{4}{b} - \tfrac{1}{2}) + \tfrac{3}{16} b^4 (\log. \tfrac{4}{b} - \tfrac{13}{12}) + \&c.$$

which we have demonstrated elsewhere, and which leads very expeditiously to a result. It is therefore convenient to have similar formulas for calculating A and B in the case of b very little; these formulas may be deduced from the value of E1 and other known relations, but they may be found directly as follows.

53. From the values of A and B developed in art. 49, these two equations may be found,

$$B = A - 2\left(\frac{1-c^2}{c}\right)\frac{dA}{dc}, \quad c(1-c^2)\frac{dA}{dc} = \int Acdc.$$

The first shews how B may be found from A; the differential of the second being taken, we get

$$(1-c^2)\frac{ddA}{dc^2} + \frac{1-3c^2}{c} \cdot \frac{dA}{dc} - A = 0 \;\ldots\ldots(\gamma)$$

As we wish to develope A according to the powers of b, we must substitute in this equation $\sqrt{(1-b^2)}$ instead of c and it will become

$$(1-b^2)\frac{ddA}{db^2} + \frac{1-3b^2}{b} \cdot \frac{dA}{db} - A = 0 \;\ldots\ldots(\delta)$$

It therefore happens, which is very remarkable, that the equation has absolutely the same form in respect of b that it has in respect of c. Hence we may conclude that we shall satisfy the equation (δ) by putting in place of A the value $A' = 1 + \frac{1}{4}b^2 + \frac{1 \cdot 9}{4 \cdot 16}b^4 + \&c.$ This value A′ is not that which we seek (seeing that it answers to the complement of the modulus), but it leads to it. Let $A = A'p$, the substitution in equation (δ) gives, after the usual reductions, $p = \alpha\int\frac{db}{b(1-b^2)A'^2}$ This value being developed into a series, and the constant quantity α determined by the known value of E1, we have

$$\frac{\pi}{2}A = A'\left[\log.\frac{4}{b} - \frac{1}{4}b^2 - \frac{13}{128}b^4 - \frac{23}{384}b^6 - \&c.\right]\ldots(\epsilon)$$

a formula which converges very fast, when b is small, and which may be conveniently employed when c is greater than *sin* 80°, or b less than *sin* 10°. In the construction of a table we have this advantage, that A′ is already calculated for a small angle when A has been calculated for its complement. For the object of these approximations it is not necessary to continue the series (ϵ) farther; however, the form of the expression being known, it is easy to find this law from equation (γ).

$$\frac{\pi}{2}A = \log.\frac{4}{b} + \tfrac{1}{4}b^2\left(\log.\frac{4}{b} - 1\right) + \frac{1 \cdot 9}{4 \cdot 16}b^4\left(\log.\frac{4}{b} - 1 - \frac{1}{2 \cdot 3}\right)$$
$$+ \frac{1 \cdot 9 \cdot 25}{4 \cdot 16 \cdot 36}b^6\left(\log.\frac{4}{b} - 1 - \frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 5}\right) + \&c. \;\ldots\ldots(\xi)$$

where it must be observed that the fractions $\frac{1}{2 \cdot 3}$, $\frac{1}{3 \cdot 5}$, are followed by $\frac{1}{4 \cdot 7}$, $\frac{1}{5 \cdot 9}$, &c. When A is known, we have B either by the formula

$$\frac{\pi}{2}B = \left(A' + 2b\frac{dA'}{db}\right)\left(\log.\frac{4}{b} - \frac{1}{4}b^2 - \frac{13}{128}b^4 - \&c.\right) - \frac{2}{(1-b^2)A'} \cdot\cdot (\gamma)$$

(in which, instead of $A' + 2b\frac{dA'}{db}$, we may put $\frac{A' - B'b^2}{1-b^2}$, or its developed value $1 + \frac{1}{4} \cdot 5b^2 + \frac{1 \cdot 9}{4 \cdot 16}9b^4$ &c.), or else by the series

$$\frac{\pi}{2}B = -2A' + \log.\frac{4}{b} + \frac{1}{4} \cdot 5b^2\left(\log.\frac{4}{b} - 1\right) + \frac{1 \cdot 9}{4 \cdot 16}9b^4 \times \left(\log.\frac{4}{b} - 1 - \frac{1}{2 \cdot 3}\right) + \&c.$$

The other co-efficients C, D, &c. may be calculated *by* the equations (ε) which apply with certainty and convenience to the case in question: It also appears that the combination of equations (η) and (ε) leads directly to the equation $\frac{4}{\pi AA'} = 1 - \frac{B}{A}c^2 - \frac{B'}{A'}b^2$, which is the theorem of art. 37.

Finally, to come to an equation of another kind, if we consider the series of functions F1, which answer to the series of *moduli* . . . c'', c', c, c°, c°, . . . of which the mean $c = \sin 45°$, it will easily appear, that the *moduli* of the functions F' and F' are complements of one another, and the same is true of F'' and $F^{\circ\prime}$, &c. besides, we have $F = 2F^{\circ}$, $F'' = 4F'$, &c. Hence, and from the preceding formula, it follows that π is equal to the limit of the quantities $\log.\frac{4}{c^{\circ}}$, $\frac{1}{2}\log.\frac{4}{c}$, $\frac{1}{4}\log.\frac{4}{c'}$, &c. and, after the 6th term, this approximation is true to at least 100 decimal places.

Of the surface of the Oblique Cone.

54. Let the radius of the base $= 1$, the height of the cone $= f$, the distance of the bottom of the perpendicular from the centre of the base $= g$, the angle at the centre, contained by the line g and any radius whatever, $= \omega$, the part of the surface of the cone which corresponds to the angle ω will be

$$c = \int \tfrac{1}{2}d\omega \sqrt{[f^2 + (1 - g\cos\omega)^2]}.$$

To reduce this integral to the common form, let us make

$\cos\omega = \frac{m + \cos\varphi}{1 + m\cos\varphi}$, or $\tan\frac{1}{2}\omega = \sqrt{\left(\frac{1-m}{1+m}\right)}\tan\frac{1}{2}\varphi$, and let us take the indeterminate quantity m, such that $f^2m + (1 - mg)(m - g) = 0$, which gives

$$m = \frac{1 + f^2 + g^2}{2g} - \sqrt{\left[\left(\frac{1 + f^2 + g^2}{2g}\right)^2 - 1\right]},$$

let us further make $c^2 = \frac{(g-m)^2}{f^2 + (g-m)^2}$, and we shall have the transformed equation

$$c = \tfrac{1}{2}(1-m)^{\frac{3}{4}}\sqrt{\left(\frac{g}{m}\right)}\int\frac{d\varphi\sqrt{(1 - c^2\sin^2\varphi)}}{(1 + m\cos\varphi)^2};$$

a formula where m is less than unity as it ought to be, in order that φ may be real, and where it may be remarked that $\sqrt{\left(\frac{g}{m}\right)}$ is the arithmetical mean between the greatest and least distance of the vertex from the circumference of the base.

If, in order to abridge, we now make $c = \frac{1}{2}(1 - m^2)^{\frac{1}{2}} \times Z\sqrt{\left(\frac{g}{m}\right)}$ we shall have

$$Z = \frac{-m\Delta\sin\varphi}{1 + m\cos\varphi} - \int\frac{m\,d\varphi\cos\varphi}{(1 - m^2\cos^2\varphi)\Delta} + \int\frac{d\varphi}{(1 - m^2\cos^2\varphi)\Delta} - \int\frac{c^2\sin^2\varphi\,d\varphi}{\Delta}.$$

Of these three integrals, the first is reducible to the arch of a circle; for let $\tan\xi = \frac{c\sin\varphi}{\Delta\sqrt{\left(1 - \frac{m}{g}\right)}}$, the integral will be $\frac{\xi}{\sqrt{(1-m^2)}\cdot\sqrt{\left(\frac{g}{m}\right)}}$. If again we make $\frac{m^2}{1 - m^2} = n$, a positive quantity, the two other integrals may be put under this form.

$$\int\frac{n - n\sin^2\varphi}{1 + n\sin^2\varphi}\cdot\frac{d\varphi}{\Delta} + \int\Delta\,d\varphi.$$

Therefore, as usual, let $E = \int\Delta\,d\varphi$, $H = \int\frac{n - n\sin^2\varphi}{1 + n\sin^2\varphi}\cdot\frac{d\varphi}{\Delta}$, and we have

$$Z = \frac{-m\Delta\sin\varphi}{1 + m\cos\varphi} - \frac{\xi\sqrt{m}}{\sqrt{(1 - m^2)}\sqrt{(g)}} + E + H.$$

In the case of the entire cone, we have simply $Z = 4E1 + 4H1$.

The quantity H is an elliptic function of the third kind, which cannot be reduced, in general, to functions of the second; thus the surface of an oblique cone is a transcendental superior to elliptic arcs, but it is comprehended in elliptic transcendentals, and consequently enjoys their properties, that is, an infinite number of portions of the surface may be compared with each other in such a manner, that their differences combined are determinable by circular arcs; for since n is positive, logarithms do not enter into these comparisons.

An infinity of oblique cones may be assigned, the surfaces of which are determinable by elliptic arcs; that this may be the case, n must be equal to one of the functions of c, for then we have found that the function H may be depressed to a lower degree. As n ought to be positive, the most simple case is that of $n = c$, which gives $g = \surd(c + c^2)$, $f = \surd(c - c^2)$; so that if we have between the lines f, g, and the radius of the base 1 or r, the equation

$$(f^2 + g^2)^2 = 2r^2 (g^2 - f^2),$$

the surface of the cone, either whole or in part, may be determined by elliptic arcs. For example, it will be found that

$$\text{the whole surface} = \frac{\pi}{2} + 2E1 - (1 - c)\,F1.$$

The developement of the surface of an oblique cone upon a plane produces a sector, the angle of which is determined by the formula.

$$\Sigma = \int \frac{d\omega \surd[f^2 + (1 - g \cos \omega)^2]}{1 + f^2 + g^2 - 2g \cos \omega},$$

or by the transformed equation

$$\Sigma = \surd\left(\frac{g}{m}\right) \cdot (1 - m^2)^{\frac{1}{2}} \int \frac{\Delta\, d\varphi}{1 + n \sin^2\varphi}.$$

This quantity therefore depends always on the same transcendentals, but it is a little more simple, because it contains only one term which belongs to elliptic functions of the third kind. In the particular case of which we have spoken, the whole sector is $\pi + 2(1 - c)\,F1$.

Of some general Formulas which may be reduced to Elliptic Functions.

The nature of elliptic transcendentals being investigated and known, it is of advantage to reduce as great a number of integral formulas to these quantities as possible. As these are infinite in number, and as various as the substitutions which may be employed, we shall be satisfied with indicating some cases where this reduction takes place.

1°. *The integral* $\int \frac{Pdx}{\sqrt{(\alpha + \beta x^2 + \gamma x^4 + \delta x^6)}}$, *in which* P *is a rational value of* x, *may be reduced to elliptic functions.*

For if we make $x^2 = z$, we may suppose $P = M + N\sqrt{z}$, M and N being rational functions of z, and the integral proposed will become

$$\int \frac{\frac{1}{2}Mdz}{\sqrt{(\alpha z + \beta z^2 + \gamma z^3 + \delta z^4)}} + \int \frac{\frac{1}{2}Ndz}{\sqrt{(\alpha + \beta z + \gamma z^2 + \delta z^3)}},$$

both which are comprehended in elliptic formulas.

2°. *Every function* $\int \frac{Pdx}{\sqrt[4]{(\alpha + \beta x^2 + \gamma x^4)}}$, *in which* P *is a rational function of* x, *may be reduced to elliptic functions.*

For we may always make $P = M + Nx$, M and N being rational, and even functions of x. Let us consider the part $\frac{Nxdx}{\sqrt[4]{(\alpha + \beta x^2 + \gamma x^4)}}$, and let us make $\sqrt[4]{(\alpha + \beta x^2 + \gamma x^4)} = z$, which gives

$$x^2 = \frac{-\beta + \sqrt{(\beta^2 - 4\alpha\gamma + 4\gamma z^4)}}{2\gamma}.$$

It is evident that by the substitution of the value of x^2, $Nxdx$ will not contain any other radical besides $\sqrt{(\beta^2 - 4\alpha\gamma + 4\gamma z^4)}$; therefore the whole difficulty is reduced to the integration of a quantity of the form

$$\frac{Qz^2dz}{\sqrt{(\beta^2 - 4\alpha\gamma + 4\gamma z^4)}},$$

in which Q is a rational function of z^4. With respect to the part $\frac{Mdx}{\sqrt[4]{(\alpha + \beta x^2 + \gamma x^4)}}$; if we make $\sqrt[4]{(\alpha + \beta x^2 + \gamma x^4)} = xy$ we shall have

$$x^2 = \frac{-\beta + \sqrt{(\beta^2 - 4\alpha\gamma + 4\alpha y^4)}}{2(\gamma - y^4)}$$

$$\frac{dx}{xy} = \frac{2x^2y^2dy}{\sqrt{(\beta^2 - 4\alpha\gamma + 4\alpha y^4)}},$$

from which it appears that the transformed expression composed of y contains also a part entirely rational, and one of the form

$$\frac{Ry^2dy}{\sqrt{(\beta^2 - 4\alpha\gamma + 4\alpha y^4)}},$$

R being a rational function of y^4. Therefore the proposed formula is always reducible to elliptic functions.

3°. *The formula* $\int \mathrm{P}dx \sqrt[4]{(\alpha + \beta x^2 + \gamma x^4)}$ *may be resolved absolutely in the same manner,* P *being any rational function of* x.

These two cases comprehend the formula $\int \mathrm{P}dx\, (\alpha + \beta x^2 + \gamma x^4)^{\pm\frac{1}{4}}$ and also the formula $\int \mathrm{Q}dy\, (\alpha + \beta y + \gamma y^2)^{\pm\frac{1}{4}}$, where Q is a rational function of y; for making $y = x^2$, this last is reduced to the others.

4°. *The formula* $\int \mathrm{P}dx\, (\alpha + \beta x + \gamma x^2 + \delta x^3)^{\pm\frac{1}{3}}$, *when* P *is a rational function of* x, *may be reduced to elliptic functions.*

This reduction may be effected in various ways. In the first place we may make one or other of the suppositions

$$\sqrt[3]{(\alpha + \beta x + \gamma x^2 + \delta x^3)} = \sqrt[3]{\alpha} + xz,$$
$$\sqrt[3]{(\alpha + \beta x + \gamma x^2 + \delta x^3)} = x\sqrt[3]{\delta} + z,$$

and the transformed expressions composed of z will be comprehended in elliptic functions. Either of the co-efficients α and δ under the radical may also be made to disappear by making $x = m + y$, or $x = m + \frac{1}{y}$, and taking for m a real root of the equation $\alpha + \beta m + \gamma m^2 + \delta m^3 = 0$. Let us suppose that we have made δ to disappear in this way, then we make $\sqrt[3]{(\alpha + \beta x + \gamma x^2)} = z$, and hence

$$x = \frac{-\beta + \sqrt{(\beta^2 - 4\alpha\gamma + 4\gamma z^3)}}{2\gamma}$$

and by substituting this value in the proposed formula, all the difficulty is reduced to the integration of a differential of the form $\frac{\mathrm{Q}dz}{\sqrt{(\beta^2 - 4\alpha\gamma + 4\gamma z^3)}}$, Q being a rational function of z.

5°. *The formula* $\int \frac{\mathrm{P}dx}{\sqrt{(\alpha + \beta x + \gamma x^2 + \delta x^3 + \gamma x^4 + \beta x^5 + \alpha x^6)}}$ *in which* P *is a rational function of* x, *may be reduced to elliptic functions.*

For if we make $x^2 + 1 = xz$, this formula becomes immediately

$$\frac{\mathrm{P}x^{-\frac{3}{2}}\, dx}{\sqrt{[\alpha(z^3 - 3z) + \beta(z^2 - 2) + \gamma z + \delta]}}.$$

Next, the value of x, which is $\frac{1}{2}z \pm \frac{1}{2}\sqrt{(z^2 - 4)}$, being substituted in P, it is evident that the result will be of the form $\mathrm{M} \pm \mathrm{N}\sqrt{(z^2 - 4)}$, M and N being rational functions of z. Again, we have $x + 1 = x^{\frac{1}{2}}\sqrt{(z + 2)}$, $x - 1 = \pm x^{\frac{1}{2}} \times$

$\sqrt{(z-2)}$; hence there results $2x^{-\frac{1}{2}} = \sqrt{(z+2)} \mp \sqrt{(z-2)}$, and

$$x^{-\frac{3}{2}}dx = \frac{\frac{1}{2}dz}{\sqrt{(z+2)}} \mp \frac{\frac{1}{2}dz}{\sqrt{(z-2)}},$$

therefore, the transformed expressions composed of z have all their terms integrable by elliptic functions.

6°. *Lastly, we may integrate in like manner the formula* $\int \frac{Pdy}{\sqrt{(\beta+\gamma y^2+\delta y^4+\gamma y^6+\varepsilon y^8)}}$, P *being any rational function of* y.

For if we make $P = M + Ny$, M and N being even functions of y, and put R for the radical, the part $\frac{Mdy}{R}$ is reduced to the preceding case, by making $\alpha = 0$, and $y^2 = x$. The other part $\frac{Nydy}{R}$ may in like manner be reduced to an elliptic function by making $y^2 = x$.

EXAMPLE.

56. Let it be proposed to integrate the differential

$$d\Pi = \frac{dx}{(3-x^2)\sqrt[3]{(1-3x^2)}}.$$

We make $\sqrt[3]{(1-3x^2)} = 1 - \frac{x}{z}$ which gives $x = \frac{3z}{2}(1+z^2)$ $-\frac{3z}{2}\sqrt{(z^4+2z^2-\frac{1}{3})}$ and the transformed expression will be

$$d\Pi = \frac{\frac{1}{2}(1-z^2)dz}{(1+z^2)(3z^2-1)} - \frac{1}{2\sqrt{3}} \cdot \frac{3z^4+1}{3z^4+2z^2-1} \cdot \frac{dz}{\sqrt{(3z^4+6z^2-1)}}.$$

The first part is rational; the second would seem to require to be decomposed into elliptic functions of the third species, because of the two factors of the denominator $3z^4+2z^2-1$. But with a little attention it appears that by making $\sqrt{(3z^4+6z^2-1)}$ $= pz$, the second part of the value of $d\Pi$ becomes $-\frac{1}{3\sqrt{3}} \cdot \frac{dp}{p^2-4}$. Thus the proposed differential may be absolutely integrated without elliptic functions, and it is reducible to the rational expression

$$d\Pi = \frac{\frac{1}{2}(1-z^2)dz}{(1+z^2)(3z^2-1)} - \frac{1}{2\sqrt{3}} \cdot \frac{dp}{p^2-4}.$$

A like simplification takes place in general for the formula

$$\int \frac{(\gamma z^4 - \alpha dz)}{(\gamma z^4 + \epsilon z^2 + \alpha)\sqrt{(\alpha + \beta z^2 + \gamma z^4)}};$$

for if the radical be put $= pz$, this formula will become $\int \frac{dp}{p^2 - \beta + \epsilon}$; a reduction sufficiently remarkable, and analogous to that of art. 38 and 39.

Of Transcendents contained in the formula $\int \frac{x^{p-1}\,dx}{\sqrt[n]{(1-x^n)^{n-q}}}$; *which is supposed to be integrated from* x = o *to* x = 1.

57. Euler has considered this transcendental in his *Calculus Integralis*, cap. ix. tom. 1. and he has demonstrated a great number of its beautiful properties. Let us with this author denote the formula in question by the character $\left[\frac{p}{q}\right]$, which is convenient for comparing different formulas of this kind in which

(1) n is the same; we have in the first place $\left[\frac{p}{q}\right] = \left[\frac{q}{p}\right]$, the first property, from which it appears that the numbers p and q may be changed the one into the other. In the next place

(2) $\left[\frac{p+n}{q}\right] = \left[\frac{p}{p+q}\right] \left[\frac{p}{q}\right]$, from which it follows that the formulas $\left[\frac{p}{q}\right]$ may always be reduced to formulas of the same kind in which p and q do not exceed n; thus we may always suppose that p and q do not exceed n.

(3) In the case of $q = n$, we have exactly $\left[\frac{n}{q}\right] = \frac{1}{q}$, and in the case $p + q = n$, the integral $\left[\frac{p}{q}\right]$ depends on the circle alone, and we have generally

(4)
$$\left[\frac{n-a}{a}\right] = \left[\frac{a}{n-a}\right] = \frac{\pi}{n \sin \frac{a\pi}{n}}.$$

These two are the only cases in which the function $\left[\frac{p}{q}\right]$ can be exactly determined; all the others present so many transcendants superior to arcs of a circle, and these, for every value of n, must be reduced to the least number possible. For this purpose we have the general equation $\left[\frac{p}{q}\right]\left[\frac{p+q}{r}\right]=\left[\frac{p}{r}\right]\left[\frac{p+r}{q}\right]$ (5) which is true, whatever be the values of p, q, and r, and which furnishes several others, remarking that every expression may be reversed. By means of this equation we may, for every value of n, form a table of all the values of $\left[\frac{p}{q}\right]$ where there only remains to be determined the transcendentals $\left[\frac{n-2}{1}\right]$, $\left[\frac{n-3}{2}\right]$, $\left[\frac{n-4}{3}\right]$, &c. of which the number is $\frac{n}{2}-1$, if n is even, and $\frac{n-1}{2}$ if n is odd. We shall refer for the demonstrations to the work of Euler. Here are some other results relating to the same subject.

58. According to equation (5) we have $\left[\frac{p}{q}\right]\left[\frac{p+q}{n-p-q}\right]=\left[\frac{p}{n-p-q}\right]\left[\frac{n-q}{q}\right]$, and $\left[\frac{p}{n-p-q}\right]\left[\frac{n-q}{n-p}\right]=\left[\frac{p}{n-p}\right]\left[\frac{n}{n-p-q}\right]$. Multiplying these two equations, and putting the values known by the formulas (3) and (4) we have

$$\left[\frac{p}{q}\right]\left[\frac{n-p}{n-q}\right]=\frac{\pi\sin(p+q)\frac{\pi}{n}}{n(n-p-q)\sin\frac{p\pi}{n}\sin\frac{q\pi}{n}},\qquad(6)$$

from which it follows that the formula $\left[\frac{n-p}{n-q}\right]$ may always be determined by the formula $\left[\frac{p}{q}\right]$, of which it is in some respects the complement. In particular we have

$$\left[\frac{a}{a}\right]\left[\frac{n-a}{n-a}\right]=\frac{2\pi\cot\frac{a\pi}{n}}{n(n-2a)}.\qquad(7)$$

$$\left[\frac{n-2a}{a}\right]\left[\frac{2a}{n-a}\right]=\frac{\pi}{na\sin\frac{2a\pi}{n}}.\qquad(8)$$

Equation (5) gives also $\left[\frac{p}{q}\right]\left[\frac{p+q}{n-p}\right]=\left[\frac{p}{n-p}\right]\left[\frac{n}{q}\right]$, whence, by putting known values,

(9) $$\left[\frac{p}{q}\right]\left[\frac{p+q}{n-p}\right]=\frac{\pi}{qn\sin\frac{p\pi}{n}}.$$

From this it follows that

(10) $$\left[\frac{a}{a}\right]\left[\frac{2a}{n-a}\right]=\frac{\pi}{an\sin\frac{a\pi}{n}}.$$

and combining this result with equation (8) we have

(11) $$\left[\frac{a}{a}\right]=\left[\frac{n-2a}{a}\right]2\cos\frac{a\pi}{n}.$$

from which it appears that the formulas $\left[\frac{n-a}{n-a}\right]$, $\left[\frac{n-2a}{a}\right]$, $\left[\frac{n-a}{2a}\right]$ may be deduced immediately from $\left[\frac{a}{a}\right]$.

(59). Let us now come to transformations by which we may know more precisely the nature of these functions. The substitution of $1-x^n=y^n$ leads to equation (1); the substitution $1-x^n=x^nz^n$ gives a rational transformed expression when $p+q=n$, and thence equation (4) is deduced: in the case when n is even, and when we have $p+q=\frac{1}{2}n$, the same substitution gives the formula

(12) $$\left[\frac{\frac{1}{2}n-a}{a}\right]=\int\frac{z^{a-1}dz}{\sqrt{(1+z^n)}},$$

which should be integrated from $z=0$ to $z=\infty$.

To obtain other results, let us suppose $1-x^n=\frac{z^n}{4x^n}$, or $x^n=\frac{1}{2}\pm\frac{1}{2}\sqrt{(1-z^n)}$ and we shall have, in the case of $p=q=a$, the transformed expression $\pm\, 2^{-\frac{2a}{n}}\int\frac{z^{a-1}dz}{\sqrt{(1-z^n)}}$; and with regard to limits it must be observed that $x=0$ gives $z=0$, $x^n=\frac{1}{2}$ gives $z=1$ and $x=1$ gives $z=0$. From which it follows, that the integral ought to be taken twice from $z=0$ to $z=1$. Thus, as x may be put in place of z we have

(13) $$\left[\frac{a}{a}\right]=2^{1-\frac{2a}{n}}\int\frac{x^{a-1}dx}{\sqrt{(1-x^n)}}.$$

Hence the successive values of $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{3}{3}\right]$, &c. may be deduced under a very convenient form, and these formulas may supply the place of those employed by Euler to express all the values of $\left[\frac{p}{q}\right]$. Indeed in the table of all these values formed in the manner of Euler we easily find that the formulas $\left[\frac{n-2}{1}\right]$, $\left[\frac{n-3}{2}\right]$, &c. may be determined by an equal number of formulas $\left[\frac{1}{1}\right]$, $\left[\frac{2}{2}\right]$, $\left[\frac{3}{3}\right]$, &c. but these are now reduced to the most simple form of which they are susceptible.

It is to be remarked that the equations (7) and (13) combined give

$$\int \frac{x^{a-1}\,dx}{\sqrt{(1-x^n)}} \times \int \frac{x^{n-a-1}\,dx}{\sqrt{(1-x^n)}} = \frac{2\pi \cot \frac{a\pi}{n}}{n(n-2a)}. \tag{14}$$

If n were even and we had $p-q=\frac{1}{2}n$ the same substitution of $1-x^n=\frac{z^n}{4x^n}$ would give this formula :

$$\left[\frac{\frac{1}{2}n+a}{a}\right] = 2^{-\frac{2a}{n}} \int \frac{x^{a-1}\,dx}{\sqrt{(1-x^{\frac{1}{2}n})}} = 2^{1-\frac{2a}{n}} \int \frac{x^{2a-1}\,dx}{\sqrt{(1-x^n)}}. \tag{15}$$

And in the case of n even, we have directly, without substitution,

$$\left[\frac{a}{\frac{1}{2}n}\right] = \int \frac{x^{a-1}\,dx}{\sqrt{(1-x^n)}}. \tag{16}$$

Suppose now $1-x^n=\frac{1}{4}z^n x^{2n}$, or $x^{-n}=\frac{1}{2}+\frac{1}{2}\sqrt{(1+z^n)}$, we shall find, in the case of $p+2q=n$, this new formula

$$\left[\frac{n-2a}{a}\right] = 2^{-\frac{2a}{n}} \int \frac{z^{a-1}\,dz}{\sqrt{(1+z^n)}} \tag{17}$$

which should be integrated from $z=0$ to $z=\infty$, and which supposes $a \angle \frac{1}{2}n$. This formula is not less simple than the formula (13) and may serve the same purpose, for if it be combined with equation (11) the result is

$$\left[\frac{a}{n}\right] = 2^{1-\frac{2a}{n}} \cos\frac{a\pi}{n} \int \frac{z^{a-1}\,dz}{\sqrt{(1+z^n)}}. \tag{18}$$

We must therefore have generally

$$(19)\qquad \int\frac{x^{a-1}\,dx}{\sqrt{(1-x^n)}} = \cos\frac{a\pi}{n}\int\frac{z^{a-1}\,dz}{\sqrt{(1+z^n)}};$$

the first integral taken from $x = 0$ to $x = 1$, and the second from $z = 0$ to $z = \infty$. I observe with respect to this equation that if n is odd, and we make $x = 1 - y^2$ in the first member and $z = y^2 - 1$ in the second, both are reducible to the same formula, viz.

$$\int\frac{(1-y^2)^{a-1}\,dy}{\sqrt{\left[n - \frac{n\,.\,n-1}{2}y^2 + \frac{n\,.\,n-1\,.\,n-2}{2\,.\,3}y^4 - \&c.\right]}}.$$

This integral in the first member is taken from $y = 0$ to $y = 1$, and in the second from $y = 1$ to $y = \infty$. Therefore these two parts are to each other as $\cos\frac{a\pi}{n}$ to 1.

Again, whether n be even or odd, formula (19) coming from formula (17) supposes $a < \frac{1}{2}n$, for otherwise, if we had $a =$ or $> \frac{1}{2}n$ the second member would become infinite, and in order that the equation should always hold true, it would be necessary to put it under the form

$$(20)\qquad \int\frac{x^{a-1}\,dx}{\sqrt{(1-x^n)}} = \cos\frac{a\pi}{n}\int\left\{\frac{z^{a-1}\,dz}{\sqrt{(1+z^n)}} - z^{a-\frac{1}{2}n-1}\,dz\right\}:$$

which is found by the last substitution, supposing $p + 2q = 2n$. From hence is derived the manner of finding the integral of the second member; for by means of equations (14) and (19) we find, a being $\not> \frac{1}{2}n$,

$$\int\frac{z^{n-a-1}\,dz}{\sqrt{(1+z^n)}} \times \int\left\{\frac{z^{a-1}\,dz}{\sqrt{(1+z^n)}} - z^{a-\frac{1}{2}n-1}\,dz\right\}$$

$$(21)\qquad = \frac{4\pi}{n(2a-n)\sin\frac{2a\pi}{n}}.$$

59. When n is uneven, the number of transcendentals necessary to determine all the formulas $\left[\frac{p}{q}\right]$ cannot be reduced below $\frac{n-1}{2}$; when n is even that number may be reduced to $\frac{n}{4}$ or $\frac{n-2}{4}$.

In short, by the combination of equations (12) and (18) we find

$$\left[\frac{a}{a}\right] = 2^{1-\frac{2a}{n}} \cos\frac{a\pi}{n}\left[\frac{\frac{1}{2}n-a}{a}\right]: \tag{22}$$

But the formula $\left[\frac{\frac{1}{2}n-a}{a}\right]$, which is the same thing as $\left[\frac{a}{\frac{1}{2}n-a}\right]$, presents only $\frac{n}{4}$, or $\frac{n-2}{4}$ different values.

This last equation gives, by putting $\frac{1}{2}n-a$ in place of a,

$$\left[\frac{\frac{1}{2}n-a}{\frac{1}{2}n-a}\right] = 2^{\frac{2a}{n}} \sin\frac{a\pi}{n}\left[\frac{a}{\frac{1}{2}n-a}\right],$$

and thus we have directly

$$\left[\frac{a}{a}\right] = 2^{1-\frac{4a}{n}} \cot\frac{a\pi}{n}\left[\frac{\frac{1}{2}n-a}{\frac{1}{2}n-a}\right], \tag{23}$$

hence it appears that it is sufficient to have the values of $\left[\frac{a}{a}\right]$ in those cases only in which a does not exceed $\frac{1}{4}n$.

If in equation (12) we put $\frac{1}{2}n-a$ in place of a, we have

$$\int\frac{z^{a-1}\,dz}{\sqrt{(1+z^n)}} = \int\frac{z^{\frac{1}{2}n-a-1}\,dz}{\sqrt{(1-z^n)}}, \tag{24}$$

an equation which besides is verified immediately by putting $\frac{1}{z}$ in place of z, in the first member: It always supposes $a < \frac{1}{2}n$. From thence, and equation (19) we deduce this other equality, which is not so evident

$$\int\frac{x^{a-1}\,dx}{\sqrt{(1-x^n)}} = \cot\frac{a\pi}{n}\int\frac{x^{\frac{1}{2}n-a-1}\,dx}{\sqrt{(1-x^n)}}. \tag{25}$$

These general results throw great light upon the particular examples (*see Euler's work already quoted*, art 392, *et seq.*)

EXAMPLE I.

60. If $n = 3$, the only transcendental required is

$$\left[\frac{a}{1}\right] = 2^{\frac{1}{3}}\int\frac{dx}{\sqrt{(1-x^3)}} = 2^{-\frac{2}{3}}\int\frac{dz}{\sqrt{(1+z^3)}},$$

and its value, which has already been found in art. 35, is

$$\left[\frac{1}{1}\right] = \frac{2\sqrt[3]{2}}{3\sqrt[4]{3}}\,\mathrm{F}^1c = \frac{\sqrt[3]{16}}{\sqrt[4]{27}}\,\mathrm{F}^1b,$$

the modulus c being sin 75°, or b being sin 15°.

EXAMPLE II.

If $n = 4$, the only transcendental necessary is

$$\left[\frac{1}{1}\right] = 2^{\frac{1}{2}}\int\frac{dx}{\sqrt{(1-x^4)}} = \int\frac{dz}{\sqrt{(1+z^4)}},$$

of which the value is F1c, c being $\frac{1}{\sqrt{2}}$ or sin 45°.

EXAMPLE III.

The case of $n = 5$ presents the two transcendentals

$$\left[\frac{1}{1}\right] = 2^{\frac{2}{5}}\int\frac{dx}{\sqrt{(1-x^5)}} = 2^{\frac{3}{5}}\cos\frac{\pi}{5}\int\frac{dz}{\sqrt{(1+z^5)}},$$

$$\left[\frac{2}{2}\right] = 2^{\frac{2}{5}}\int\frac{xdx}{\sqrt{(1-x^5)}} = 2^{\cdot}\cos\frac{2\pi}{5}\int\frac{zdz}{\sqrt{(1+z^5)}};$$

but they do not appear reducible to elliptic functions.

EXAMPLE IV.

The case of $n = 6$ presents the single transcendental

$$\left[\frac{1}{1}\right] = 2^{\cdot}\int\frac{dx}{\sqrt{(1-x^6)}} = 2^{\frac{2}{3}}\cos\frac{\pi}{6}\int\frac{dz}{\sqrt{(1+z^6)}}.$$

The first form by making $\frac{1}{x^2} = 1 + z^2$ gives for a result $\left[\frac{1}{1}\right] = \frac{\sqrt[3]{4}}{\sqrt[4]{3}}$. F1$b$, the modulus b being sin 15°. The second form by making $\frac{1}{z^2} = y^2 - 1$ gives for a result $\left[\frac{1}{1}\right] = 2^{\frac{2}{3}}\cos\frac{\pi}{6}$ F1c, c being sin 75°. These two results ought to agree, we must therefore have F1c = $\sqrt{3}$ F1b, as we have found in art. 36. The transcendental $\left[\frac{1}{1}\right]$ being thus known, we may deduce from it the two which Euler has employed to determine all the others; they are

$$\left[\frac{4}{1}\right] = \frac{1}{\sqrt{3}}\left[\frac{1}{1}\right],\quad \left[\frac{3}{2}\right] = \frac{2^{-\frac{2}{3}}}{\sqrt{3}}\left[\frac{1}{1}\right].$$

EXAMPLE V.

The case of $n = 8$ presents the two transcendentals

$$\left[\frac{1}{1}\right] = 2^{\frac{3}{2}}\int\frac{dx}{\sqrt{(1-x^8)}} = 2^{\frac{3}{4}}\cos\frac{\pi}{8}\int\frac{dz}{\sqrt{(1+z^8)}},$$

$$\left[\frac{2}{2}\right] = 2^{\frac{1}{3}}\int\frac{dx}{\sqrt{(1-x^8)}} = 2^{\frac{1}{2}}\cos\frac{\pi}{4}\int\frac{zdz}{\sqrt{(1+z^8)}}.$$

The second, by putting x^2 instead of x, becomes $2^{-\frac{1}{2}}$ $\int\frac{dx}{\sqrt{(1-x^4)}}$ and thus its value is $\left[\frac{2}{2}\right]=\frac{1}{2}\mathrm{F}1\left(\frac{1}{\sqrt{2}}\right)$. To value the formula $\left[\frac{1}{1}\right]$ we shall choose the second form, and making $1+z^4=pz^2$, which gives $z=\frac{1}{2}\sqrt{(p+2)}\pm\frac{1}{2}\sqrt{(p-2)}$, we have by substitution

$$\frac{dz}{\sqrt{(1+z^8)}}=\frac{\frac{1}{4}dp}{\sqrt{(p+2)}\,.\,\sqrt{(p^2-2)}}\mp\frac{\frac{1}{4}dp}{\sqrt{(p-2)}\sqrt{(p^2-2)}}.$$

These two parts ought to be twice integrated, from $p=2$ to $p=\infty$, and as in each interval they have contrary signs, it is sufficient to take the double of the greatest part, and to integrate $\frac{\frac{1}{4}dp}{\sqrt{(p-2)}\,.\,\sqrt{(p^2-2)}}$ from $p=2$ to $p=\infty$. Let $p=2+q^2$, we have then

$$\int\frac{dz}{\sqrt{(1+z^8)}}=\int\frac{dq}{\sqrt{(q^2+2-\sqrt{2})}\,.\,\sqrt{(q^2+2+\sqrt{2})}},$$

the limits of q being 0 and ∞. Let us, to abridge, put $m=\sqrt{(2+\sqrt{2})}=2\cos\frac{\pi}{8}$, $q=m\cot\varphi$, $c^2=2\sqrt{2}-2$, and the expression is transformed to $\frac{1}{m}\,\frac{d\varphi}{\sqrt{(1-c^2\sin^2\varphi)}}$, of which the integral between the required limits is $\frac{1}{m}\,\mathrm{F}1c$. It may be remarked that the modulus c has for its complement $b=\sqrt{2}-1$, so that the elliptic function which occurs in this result is that which we have already considered in art. 37, and which has peculiar properties. Hence we have

$$\left[\frac{1}{1}\right]=\frac{1}{\sqrt[4]{2}}\mathrm{F}1c=\sqrt[4]{2}\,\mathrm{F}1b\;(b=\sqrt{2}-1).$$

If we now make $\left[\frac{1}{1}\right]=\mathrm{M}$, $\left[\frac{2}{2}\right]=\mathrm{N}$, we shall from these known values derive those of the three transcendentals which are required by Euler's method to form the table of the values of $\left[\frac{p}{q}\right]$, and we have

$$\left[\frac{6}{1}\right]=\frac{\mathrm{M}}{2\cos\frac{\pi}{8}},\quad\left[\frac{5}{2}\right]=\frac{2^{-\frac{1}{4}}\mathrm{N}}{\cos\frac{\pi}{8}},\quad\left[\frac{4}{3}\right]=\frac{\sqrt{2}-1}{\sqrt[4]{8}}\,\mathrm{M}.$$

EXAMPLE VI.

The case of $n=10$ depends on two transcendentals only,

$$\left[\frac{4}{1}\right] = \int \frac{dz}{\sqrt{(1+z^{10})}}, \quad \left[\frac{3}{2}\right] = \int \frac{zdz}{\sqrt{(1+z^{10})}}$$

For, according to equation (22), these two values are sufficient to determine all those of $\left[\frac{a}{a}\right]$. But $\left[\frac{4}{1}\right]$, or $\left[\frac{1}{4}\right]$ is also represented by $\int \frac{z^3dz}{\sqrt{(1+z^{10})}}$, and by putting z^2 instead of z in this value, as well as in that of $\left[\frac{3}{2}\right]$, the two transcendentals in question are

$$\left[\frac{4}{1}\right] = \tfrac{1}{2}\int \frac{zdz}{\sqrt{(1+z^5)}}, \quad \left[\frac{3}{2}\right] = \tfrac{1}{2}\int \frac{dz}{\sqrt{(1+z^5)}}.$$

These are the same as in the case of $n = 5$. In general, the case of $n = 4m+2$ is reducible always to that of $n = 2m+1$.

EXAMPLE VII.

Lastly, let us examine the case of $n = 12$ as giving occasion to remarkable reductions. This case depends on three formulas

$$\left[\frac{5}{1}\right] = \int \frac{dz}{\sqrt{(1+z^{12})}}, \quad \left[\frac{4}{2}\right] = \int \frac{zdz}{\sqrt{(1+z^{12})}}, \quad \left[\frac{3}{3}\right] = \int \frac{z^2dz}{\sqrt{(1+z^{12})}}.$$

The formula $\left[\frac{4}{2}\right]$ or $\left[\frac{2}{4}\right]$ is also equal to $\int \frac{z^3dz}{\sqrt{(1+z^{12})}}$, and by putting z instead z^4 it is reduced to $\tfrac{1}{4}\int \frac{dz}{\sqrt{(1+z^3)}}$.

In like manner the formula $\left[\frac{3}{3}\right]$, by putting z instead of z^3, is reduced to $\tfrac{1}{3}\int \frac{dz}{\sqrt{(1+z^4)}}$; these values being known, we have immediately

$$\left[\frac{4}{2}\right] = \tfrac{1}{4}\int \frac{dz}{\sqrt{(1+z^3)}} = \frac{1}{\sqrt[4]{27}}\,F1\,(c = \sin 15^{\circ})$$

$$\left[\frac{3}{3}\right] = \tfrac{1}{3}\int \frac{dz}{\sqrt{(1+z^4)}} = \tfrac{1}{3}F1\,(c = \sin 45^{\circ}).$$

It remains to determine the formula $\frac{dz}{\sqrt{(1+z^{12})}}$. To do this let $1+z^4 = pz^2$, and the differential to be integrated will become

$$\frac{\frac{1}{4}dp}{\sqrt{(p^3-3p)}} \pm \frac{\frac{1}{4}dp\sqrt{p}}{\sqrt{(p^2-8)}\cdot\sqrt{(p^2-4)}}.$$

and we shall find, as above, that the whole difficulty consistsin integrating the double of the second part from $p = 2$ to $p = \infty$, Let $p = q^2$; we have then

$$\int \frac{dz}{\sqrt{(1 + z^{12})}} = \int \frac{q^2 dq}{\sqrt{(q^8 - 7q^4 + 12)}}.$$

If we now make $m^4 = 12$, and $q^4 + m^2 = q^2 y$, which gives $2q = \sqrt{(y + 2m)} + \sqrt{(y - 2m)}$, we shall have for the transformed expression

$$\int \frac{\frac{1}{4}dy}{\sqrt{(y+2m)}\sqrt{(y^2 - 2m^2 - 7)}} + \int \frac{\frac{1}{4}dy}{\sqrt{(y-2m)}\sqrt{(y^2 - 2m^2 - 7)}}$$

the two parts of which ought to be integrated from $y = 2 + \sqrt{3}$, or $y = 2m^2 + 7$ to $y = \infty$. Let $2 + \sqrt{3} = n$, $y + 2m = x^2$, the first part becomes

$$\int \frac{\frac{1}{2}dx}{\sqrt{(x^2 - 2m - n)}\sqrt{(x^2 - 2m + n)}}.$$

Now let $x^2 = \frac{2m + n}{\cos^2\varphi}$, and $c = \frac{n - 2m}{2n}$, or $c = \frac{\sqrt[4]{4} - \sqrt[4]{3}}{1 + \sqrt{3}}$

and the last expression will be $\frac{1}{2\sqrt{2n}} \int \frac{d\varphi}{\sqrt{(1 - c^2 \sin^2\varphi)}}$, the value of which between the required limits is $\frac{1}{2\sqrt{2n}}$ F1c, or $\left(\frac{\sqrt{3} - 1}{4}\right)$ F1c.

In like manner, if we make $b = \frac{\sqrt[4]{4} + \sqrt[4]{3}}{1 + \sqrt{3}}$, the integral of the second part will be $\left(\frac{\sqrt{3} - 1}{4}\right)$ F1b; and we may design the second modulus by b, because, we have $b^2 + c^2 = 1$, and so they are the complements of each other. We have therefore,

$$\left[\frac{5}{1}\right] + \int \frac{dz}{\sqrt{(+1z^{12})}} = \left(\frac{\sqrt{3} - 1}{4}\right) (\text{F}1b + \text{F}1c).$$

But it happens in regard to these functions, that we have exactly F1b = 3F1c, the value of $\left[\frac{5}{1}\right]$ is therefore reducible to this $\left[\frac{5}{1}\right] = (\sqrt{3} - 1)$ F1c, and as c is very small, this function may be easily valued. Hence it appears that the case of $n = 12$ is also resolved completely by functions of the first kind.

The end of Le Gendre's paper on Elliptic Trancendentals.

ARTICLE II.

To the Editor of the Repository.

Sir,

As in one of your earlier numbers you inserted the following formula for the length of an arc, $q = \sec.\frac{q}{2} \times \sec.\frac{q}{4} \times \sec.\frac{q}{8} \times$ &c. I hope you will have no objection to insert the following geometrical approximation. It is given by Euler, Novi Comment. Petropol. Vol. III.—q is taken = a quadrant for facility of application in practice.

I am, &c.

R. I. Dishneagh.

Let AB (fig. 10, pl. A′.) represent a quadrantal arc: rad. $= 1$, Bisect it in c, and let OC be the secant and $= \therefore$ to secant $\frac{q}{2}$. Bisect Ac in d and draw OdD to meet CD $\perp$ to OC, $\therefore$

$$\text{OD} : \text{OC} :: \sec \text{COD} : 1$$
$$:: \sec \frac{q}{4} : 1$$

$\therefore \text{OD} = \text{OC} \times \sec\frac{q}{4} = \sec\frac{q}{2} \times \sec\frac{q}{4}$. In the same way bisecting Ad in e and making a similar construction.

$\text{OE} = \text{OD} \times \sec\frac{q}{8} = \sec\frac{q}{2} \times \sec\frac{q}{4} \times \sec\frac{q}{8}$. Hence bisecting A$e$ in f, Af in g, &c. and making the same construction in each case.

$$\text{OF} = \text{OE} \times \sec\frac{q}{16} = \sec\frac{q}{2} \times \sec\frac{q}{4} \times \sec\frac{q}{8} \times \sec\frac{q}{16}.$$
$$\text{OG} = \&c.$$
$$\ldots\ldots = \ldots\ldots$$

And ultimately,

$\text{OZ} = \sec\frac{q}{2} \times \sec\frac{q}{4} \times \sec\frac{q}{8} \times \sec\frac{q}{16} \times$ &c., $\therefore$ OZ $= q$ accurately.

End of the third Part of the third Volume.

Printed by W. Glendinning, 25, Hatton Garden, London.

y who
nually
iission
1811.
ence,

posed

years
under
f the

nation

metic
liffer.

their
h the

$= w$;
ecific

, also
nuth,
angle

latus
locity

T

As i

formul

x &c.
geomet
ment.
ty of a

Let

Bisect

Bisect

∴ o

bisecti

(

bisecti
tion in

(

(

...

And

o

$= q$ a

1

CAMBRIDGE PROBLEMS.

As there are many readers of the Mathematical Repository who would, no doubt, be glad to see the problems proposed annually at the University of Cambridge, the Editor has obtained permission from the Moderators to insert in his work those for the year 1811. And if in time coming he shall meet with the same indulgence, he will continue to insert those of future years.

It is, however, to be observed, that they are not proposed here for the purpose of being resolved.

The problems that have been proposed for the last ten years have been lately collected into a volume and published under the title of *Cambridge Problems*, by the Publisher of the "University Calendar."

THE SENATE-HOUSE PROBLEMS,

Given to the Candidates for Honours during the Examination for the Degree of B. A. in January, 1811,

BY THE TWO MODERATORS,

MONDAY, JAN. 14, 1811.

MORNING PROBLEMS.—Mr. TURTON.

1. The interior angles of a rectilinear figure are in arithmetic progression; the least angle is 120°, and the common difference 5°. Required the number of sides.

2. Given the radii of two spheres, and the line joining their centres; Find, in that line, the position of an eye, to which the apparent surfaces will together be the greatest possible.

3. The weight of a globe in air $= W$, and in water $= w$; Find its diameter and specific gravity, having given the specific gravity of water (S) and of air (s).

4. Having given the latitude of the place, the day and hour, also the latitude and longitude of a star; Find its altitude and azimuth, the point where its vertical circle cuts the ecliptic, and the angle which they make.

5. Find the ratio of the velocity at the extremity of the latus rectum of an ellipse (the force being in the focus) to the velocity

in a circle whose radius is the distance of the nearer apside from the focus; and shew that, as the excentricity is increased, this ratio approaches to a ratio of equality.

6. Shew that the spaces described by a body, impelled from rest by a finite variable force, are, "ipso motûs initio," in the duplicate ratio of the times.

7. If, to the radius unity, A = the sum of the tangents of any number of arcs; B = the sum of the products of every two of them: C = the sum of the products of every three; and so on: Shew that the tangent of the sum of those arcs will be

$$\frac{A-C+E-G+\&c.}{1-B+D-F+\&c.}$$

8. Shew that the fluent of $\frac{d\dot{z}}{z\sqrt{a+bz^n}} = \frac{2d}{n\sqrt{a}} \times$ hyp log. $\frac{\sqrt{a+bz^n}-\sqrt{a}}{\sqrt{bz^n}}$; and find the fluent of $\frac{z^{\frac{1}{2}n-1}\dot{z}}{z^{pn}\times\overline{a+bz^n}}$, and of $\frac{\dot{z}}{z^{\frac{1}{2}n+1}\times\sqrt{a+bz^n}}$.

9. Find the value of $a \cdot \overline{a+r} \cdot \overline{a+2r}$, &c. continued to any number of factors.

10. Find the nature and length of the caustic, when the reflecting curve is a circular arc, and the focus of incident rays is in the circumference of the circle.

11. At a given place, at a given hour, and on a given day, required the point of the compass on which a rainbow would appear.

12. Given the latitude of the place, and the day of the year; Find the hour at which two stars, whose right ascensions and declinations are known, will be on the same azimuth.

13. Given the perihelion distance of a comet describing a parabola, and the radius of the earth's orbit, here supposed to be circular; Compare the time of the comet's moving through 90 degrees of true anomaly with the length of the solar year.

14. Define the center of spontaneous rotation of a system; explain the principle on which that center may be found; and shew that if the system revolve round an axis, passing through that center perpendicular to the plane of revolution, the former point of impact will become the center of percussion.

MONDAY AFTERNOON.

Fifth and Sixth Classes.—Mr. TURTON:

1. The interest of 25*l.* for 3½ years, at simple interest, was found to be 3*l.* 18*s.* 9*d.* Required the rate per cent. per annum.

2. If the first of six magnitudes be to the second as the third to the fourth, and the fifth to the second as the sixth to the fourth; Prove that the first and fifth together will be to the second, as the third and sixth together to the fourth.

3. Given $\sqrt{\dfrac{a^2+x^2}{a+x}}$, a minimum: Find the value of x.

4. Find the fluent of $\dfrac{d\dot{z}}{\sqrt{a+bz}}$, of $\dfrac{d\dot{z}}{\sqrt{a+bz^2}}$, and of $\dfrac{x\dot{x}}{\sqrt{ax-x^2}}$.

5. Find an expression for the sum of n terms of the series $\frac{1}{5}-\frac{2}{15}+\frac{4}{45}$ &c. that may be applied according as n is an even or an odd number.

6. Shew, that if any momenta be communicated to the parts of a system, its center of gravity will move in the same manner that a body, equal to the sum of the bodies in the system, would move, were it placed in that center, and the same momenta, in the same directions, communicated to it.

7. Compare the time of oscillation in a given cycloid, with the time of falling down a vertical line equal to the whole length of the cycloid.

8. Required the equation of which the roots are $\pm\sqrt{-2}$, 3, 4.

9. If a body fall through a finite altitude AS, the force varying inversely as the square of the distance, and on AS, a semicircle ADS be described; Prove that the area described by the indefinite radius SD is equal to the area uniformly described, in the same time, in a circle whose radius is the half of SA.

10. Given the latitude of the place, and the sun's declination; Find the length of the day.

11. Compare the time of descent through any space AS, the force at S varying inversely as the square of the distance, with the periodic time in a circle whose radius is SA.

12. Explain by what means the accelerating forces of bodies are compared; also, by what means their moving forces; and shew that the accelerating force varies as the moving force directly, and the quantity of matter moved inversely.

13. Prove, that if the object placed before a spherical reflector be a straight line, the image is a conic section.

14. Two weights, of which one (P) is known, are connected by a string passing over a fixed pulley; P, in descending from rest through the space s, acquires the velocity a. Find the other weight.

15. Find the variation of the force by which a body describes a parabola, round a center of force in the focus.

16. Find the actual periodic time in a given ellipse, described

round a center of force in the focus; supposing that the force at a given distance (d) is to the force of gravity as F to 1.

Third and Fourth Classes.—Mr. JEPHSON.

1. What is the interest of 115*l.* for $5\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent?

2. Given $x+y-\sqrt{\frac{x+y}{x-y}}=\frac{6}{x-y}$, $x^2+y^2=41$; Find x and y; and find x in the equation $a^x+\frac{1}{a^x}=b$.

3. Investigate the rule for extracting the square root of a binomial surd, and apply it to determine the square root of $-2m\sqrt{-m^2}$.

4. Investigate the fluxional expression for the area of curves, and apply it to find the area that lies between the asymptotes of a common hyperbola.

5. If (a) be an arithmetic, (b) a geometric, (c) a harmonic mean; Shew that (a) is greater than (b), and (b) greater than (c).

6. If any quantities, whose differences are inconsiderable in respect to the quantities themselves, be in arithmetical progresssion, the same quantities are also in geometrical progression.

7. Inscribe, in a triangle, a parallelogram similar to a given parallelogram.

8. Two balls A and B are placed on a billiard table; In what direction must A, which is perfectly elastic, be struck, that it may hit B after impinging upon two adjacent sides of the table?

9. The specific gravity of a cylinder of known length is greater than that of the fluid in which it is placed: Determine the depth of its lower surface.

10. The sine of incidence is to the sine of refraction, out of a denser medium into a rarer, as (n) to (m); Give a geometrical construction for determining the greatest possible angle of incidence.

11. Find the fluents of $\frac{\dot{x}}{x^3\sqrt{a^4+x^4}}$, and $\dot{x}\sqrt{bx-cx^2}$.

12. Sum the following series:

$$\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\text{\&c. in inf. and } \frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\text{\&c. in inf.}$$

13. The angles at the base of an isosceles spherical triangle are equal.

14. Construct a horizontal dial.

15. If the moveable orbit (NEWT. Sect IX.) be a logarithmic spiral, what is the nature of the curve traced out in fixed space?

16. Prove that one of the roots of the equation $x^3-qx-r=0$, when squared, will lie between (q) and $\frac{1}{3}q$.

Evening Problems.—Mr. JEPHSON.

1. Shew that a^n-b^n is, and that a^n+b_n is not, divisible by $a-b$. Is a_n-b_n divisible by $a+b$ (n an integer)?

2. The chord is ultimately parallel to the tangent of the middle point of the arc. Required a proof.

3. Prove that the total number of combinations of (n) things is 2^n-1; and apply the expression to find the number of different sums that may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence.

4. Define similar curves when referred to a point, and shew in what case epicycloids are similar.

5. A lever, at whose extremities P and Q hung by threads, balance each other, is made to revolve about its fulcrum; Shew that, if the threads be equal, P and Q describe concentric circles; if unequal, similar segments of circles.

6. Two equal weights balance each other by means of three tacks forming an isosceles triangle, the base of which is horizontal; Shew that the whole pressure of the tacks, estimated in the direction in which the weights act, is equal to the sum of the weights.

7. Shew that the pressure of a cylinder, infinite in height upon the earth at rest, equals the weight of another cylinder of the same base, whose length is equal to the radius of the earth.

8. Compare, *geometrically*, the resistance upon a paraboloid moving in the direction of its axis, with the resistance upon its circumscribing cylinder.

9. If two canals be cut through the earth at rest, then the times of two bodies being attracted through these canals will be equal.

10. $x^y=y^x$; Give a geometrical construction for determining two corresponding values of x and y.

11. (1.) Take the fluxion of the two quantities x^{yz}, x^{y^z}.

(2.) Find the fluents of $\frac{\dot{x}}{(a^n+x^n)^{\frac{n+1}{2}}}$, $\frac{\dot{x}}{(x-1)^{\frac{3}{2}}.(x+\ )^{\frac{1}{2}}}$, $\dot{x}\int\dot{x}\int\frac{\dot{x}}{x}$.

(3.) Shew that $e^{\int\frac{\dot{\theta}}{\sin.\theta}}=\tan\left(\frac{\theta}{2}\right)$; (e) being the base of the hyp. logarithms.

12. Sum the following series:

$$\frac{1}{m.(m+r)}+\frac{1}{(m+r).(m+2r)}+\frac{1}{(m+2r).(m+3r)}+\text{\&c, in inf.}$$

round a center of force in the focus; supposing that the force at a given distance (d) is to the force of gravity as F to 1.

Third and Fourth Classes.—Mr. JEPHSON.

1. What is the interest of 115*l.* for $5\frac{1}{2}$ years, at $4\frac{1}{2}$ per cent?

2. Given $x+y-\sqrt{\frac{x+y}{x-y}}=\frac{6}{x-y}$, $x^2+y^2=41$; Find x and y; and find x in the equation $a^x+\frac{1}{a^x}=b$.

3. Investigate the rule for extracting the square root of a binomial surd, and apply it to determine the square root of $-2m\sqrt{-m^2}$.

4. Investigate the fluxional expression for the area of curves, and apply it to find the area that lies between the asymptotes of a common hyperbola.

5. If (a) be an arithmetic, (b) a geometric, (c) a harmonic mean; Shew that (a) is greater than (b), and (b) greater than (c).

6. If any quantities, whose differences are inconsiderable in respect to the quantities themselves, be in arithmetical progression, the same quantities are also in geometrical progression.

7. Inscribe, in a triangle, a parallelogram similar to a given parallelogram.

8. Two balls A and B are placed on a billiard table; In what direction must A, which is perfectly elastic, be struck, that it may hit B after impinging upon two adjacent sides of the table?

2. The specific gravity of a cylinder of known length is greater than that of the fluid in which it is placed: Determine the depth of its lower surface.

10. The sine of incidence is to the sine of refraction, out of a denser medium into a rarer, as (n) to (m); Give a geometrical construction for determining the greatest possible angle of incidence.

11. Find the fluents of $\frac{\dot{x}}{x^3\sqrt{a^4+x^4}}$, and $\dot{x}\sqrt{bx-cx^2}$.

12. Sum the following series:

$$\frac{1}{2.4.6}+\frac{1}{4.6.8}+\frac{1}{6.8.10}+\&c.\text{ in inf. and }\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-\&c.\text{ in inf.}$$

13. The angles at the base of an isosceles spherical triangle are equal.

14. Construct a horizontal dial.

15. If the moveable orbit (NEWT. Sect IX.) be a logarithmic [illegible] ature of the curve traced out in fixed space?

16. Prove that one of the roots of the equation $x^3 - qx - r = 0$, when squared, will lie between (q) and $\frac{1}{3}q$.

Evening Problems.—Mr. JEPHSON.

1. Shew that $a^n - b^n$ is, and that $a^n + b^n$ is not, divisible by $a - b$. Is $a^n - b^n$ divisible by $a + b$ (n an integer)?

2. The chord is ultimately parallel to the tangent of the middle point of the arc. Required a proof.

3. Prove that the total number of combinations of (n) things is $2^n - 1$; and apply the expression to find the number of different sums that may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence.

4. Define similar curves when referred to a point, and shew in what case epicycloids are similar.

5. A lever, at whose extremities P and Q hung by threads, balance each other, is made to revolve about its fulcrum; Shew that, if the threads be equal, P and Q describe concentric circles; if unequal, similar segments of circles.

6. Two equal weights balance each other by means of three tacks forming an isosceles triangle, the base of which is horizontal; Shew that the whole pressure of the tacks, estimated in the direction in which the weights act, is equal to the sum of the weights.

7. Shew that the pressure of a cylinder, infinite in height upon the earth at rest, equals the weight of another cylinder of the same base, whose length is equal to the radius of the earth.

8. Compare, *geometrically*, the resistance upon a paraboloid moving in the direction of its axis, with the resistance upon its circumscribing cylinder.

9. If two canals be cut through the earth at rest, then the times of two bodies being attracted through these canals will be equal.

10. $x^y = y^x$; Give a geometrical construction for determining two corresponding values of x and y.

11. (1.) Take the fluxion of the two quantities x^{yz}, x^{y^z}.

(2.) Find the fluents of $\frac{\dot{x}}{(a^n + x^n)^{\frac{n+1}{2}}}$, $\frac{\dot{x}}{(x-1)^{\frac{3}{2}} . (x+)^{\frac{1}{2}}}$, $\dot{x}\int\dot{x}\int\frac{\dot{x}}{x}$,

(3.) Shew that $e^{\int\frac{\dot{\theta}}{\sin.\theta}} = \tan\left(\frac{\theta}{2}\right)$; (e) being the base of the hyp. logarithms.

12. Sum the following series:

$$\frac{1}{m.(m+r)} + \frac{1}{(m+r).(m+2r)} + \frac{1}{(m+2r).(m+3r)} + \text{\&c, in inf.}$$

11. A and B begin to fall at the same time from different points in the same vertical line, and with the velocities acquired move along a horizontal plane, till one overtakes the other; Shew that the time of A's descent is equal to the time of B's uniform motion.

12. A bag contains (n) balls, A, B, C, D, &c. (p) of them are to be taken out; What is the chance of drawing (p) specified balls?

13. The equation to a catenary is $z^2 = 2ax + x^2$; a parabola, whose latus rectum $= (8a)$ is described with the same vertex and axis; Shew that any ordinate of the catenary, together with its corresponding arc, is equal to the corresponding arc of the parabola.

14. A chain whose length is (l) is placed along an inclined plane whose height is (n) and length (m), so that one end may coincide with the lowest point of the plane; Shew that the whole time of the chain's sliding off the plane is equal to

$$\sqrt{\frac{m . l}{m-n}} \times \text{hyp. log.} \left\{\frac{m+\sqrt{m^2-n^2}}{n}\right\}.$$

TUESDAY AFTERNOON.

Third and Fourth Classes.—Mr. TURTON.

1. Explain what is meant by the present worth of money due after a certain time; explain also the principle on which is founded the rule for calculating the present worth; and find the due nine months hence, discount being allowed at $4\frac{1}{2}$ per cent. present worth of 430*l.* per annum.

2. Solve the equation $3x^n \sqrt[3]{x^n} + \frac{4x^n}{\sqrt[3]{x^n}} = 4$.

3. Find the fluent of $\frac{x^2\dot{x}}{\sqrt{2ax-x^2}}$, and that of $\frac{\dot{x} \times \overline{2ax-x^2}|^{\frac{3}{2}}}{x}$.

4. Find the dimensions of the greatest cylinder that can be cut out of a solid formed by the revolution of a curve round its axis, of which the equation is $a^m x^n = y^{m+n}$, and the whole axis $= b$.

5. Given the altitude of an orifice in the side of a vessel filled with fluid, and the distance on the horizontal plane at which the fluid falls; Determine, by construction, the altitude of the vessel.

6. If $a =$ the altitude of a conic frustum, and b, c be the radii of its bases, also $p = 3.14159$ &c. then the solidity $= \frac{pa}{3} \times \overline{b^2+bc+c^2}$. Required a proof.

7. The force varying inversely as $\overline{\text{dist.}}|^{n+1}$, find the area ABFD (Newton, Prop. xxxix.) when the ordinate $=$ M at the distance d from the center; also the *fluxion* of the area ATVME, the ordinate of this curve at the distance r from the center being $=$ N.

8. Having given the latitude of the place, the sun's declination and altitude; Find his azimuth, the time of observation, and the angle of position.

9. Having measured the shadow of a tower on the horizontal plane, on a given day, at noon, in a known latitude: Shew how the altitude of the tower may be found.

10. Construct Newton's Telescope, and investigate its magnifying power.

11. Find the variation of the force tending to the focus of an hyperbola, by which the opposite hyperbola may be described.

12. Supposing the attraction of the earth and moon to be as their quantities of matter directly, and the squares of their distances inversely; having given their quantities of matter and their distance, find that point between them, at which a body would be at rest.

13. Make a body oscillate in a given epicycloid.

Fifth and Sixth Classes.—Mr. Jephson.

1. Agreed for the carriage of $2\frac{1}{2}$ tons of goods, $2\frac{9}{10}$ miles, for $\frac{3}{40}$ of a guinea; What is that per cwt. for a mile?

2. What is the amount of 120*l.* 10*s.* for $2\frac{1}{2}$ years, at $4\frac{3}{4}$ per cent.?

3. Divide .7584 by 316.; and find the sum of the recurring decimal 5.72323 &c.

4. Solve the following equations

$$\left.\begin{array}{r} x+y+\sqrt{x+y}=6 \\ x^2+y^2=10 \end{array}\right\} ; \quad \frac{x}{x+4}+\frac{4}{\sqrt{x+4}}=\frac{21}{x}.$$

5. Prove that if $A \propto B$, $A \pm B \propto B$.

6. Divide $x^3 - nax^2 + na^2x - a^3$ by $x - a$.

7. The number of combinations which (n) things admit of when taken four and four together, is to the number which they admit of when taken two and two, as 15 to 2. Required the number of quantities.

8. Define the sine of an arc; and prove that the sines are the same, drawn from either extremity of the arc.

9. Given the center of a circle; Find its diameter by means of the compasses alone.

10. Take the fluxion of $(a^2 + x^2) \cdot \sqrt{a^2 - x^2}$; and find the $\int \frac{a^2\dot{x}}{b^4 + c^2x^2}$.

11. $\frac{x^{n+1}}{2a^n} + \frac{a^n}{2x^{n-1}} =$ a maximum; Find x.

12. The sum of (n) terms of the series 1, 3, 5, 7, &c. is to the sum of ($n-1$) terms of the series 2, 4, 6, &c. as (n) to ($n-1$). Required proof.

13. Let a perfectly hard body (A) impinge upon another (B) four times as great, and at rest, with a known velocity; Find B's velocity after impact.

14. An eye is placed in the principal focus of a concave spherical reflector; Compare the apparent magnitudes of the object and image, when the former is situated half way between the focus and surface.

15. An inverted cone is filled with a fluid; Determine at what distance from the vertex a horizontal section will sustain the greatest pressure.

16. When the center of force is without the circle, find its variation in terms of the variable distance.

17. Let a body begin to fall from an infinite distance, force varying as $\frac{1}{\text{dist.}|^{2}}$: Shew that its velocity at any point of its descent is equal to the velocity that it would acquire through the remaining distance, force at that point being continued constant.

TUESDAY EVENING.

Evening Problems.—Mr. TURTON.

1. Define similar curves when referred to their axes; and prove that all parabolas are similar curves.

2. A given rectangular parallelogram is immersed vertically in a fluid, with one side coincident with the surface. From one of its angles, it is required to draw a straight line to the base, so that the pressures on the parts into which the parallelogram is divided may be in a given ratio.

3. Find that point, in the periphery of an ellipse, from which a body must fall, towards the center of force in the focus, through the greatest or least space to acquire the velocity in the curve, at the point from which it fell; and shew, from the fluxional equation, whether the point determined gives the space a maximum or minimum.

4. Construct GREGORIE's Telescope, and investigate its magnifying power.

5. The times of falling, from different altitudes, into the same center of force, vary as the n^{th} powers of those altitudes. Required the variation of the force.

6. Suppose a person to stand before a vertical plane mirror,

at any distance from it; given the altitude of the eye above the bottom of the mirror, find the part of the body that will be seen; and shew that exactly the same part will be visible at all distances from the mirror.

7. If parallel rays be incident on a sphere of given refracting power, find that ray, of which when produced, the part included by the sphere will be to the part included of the refracted ray, in a given ratio.

8. On a given day, at a given hour, and in a given latitude, it is required to find the length and direction of the shadow of a stick of given length, inclined to the horizon at a given angle, and in a given direction.

9. Resolve $\frac{1}{1-2lz+z^{2n}}$ into quadratic factors, when l is less than unity.

10. In the expansion of $\overline{a+b+c+d+\&c.}|^n$, investigate the coefficient of the term involving the literal product $a^p\ b^q\ c^r\ d^s$ &c.

12. Given $a\dot{x}+\overline{bx+cy}.\dot{x}=d\dot{y}+\overline{bx+cy}.n\dot{x}$; Find the relation between x and y.

12. Sum the following series: viz.

$\frac{3}{1^2 \cdot 2^2}+\frac{5}{2^2 \cdot 3^2}+\frac{7}{3^2 \cdot 4^2}+$ &c. to n terms, and ad inf.

$\frac{1}{1.2.3.4.5}+\frac{4}{4.5.6.7.8}+\frac{9}{7.8.9.10.11}+$ &c. to n terms, and ad inf.

13. Find the fluents of $\dot{x}\int\frac{\dot{x}}{1-x^3}$, and of $\frac{z^{2n-1}\dot{z}}{g+hz^n\sqrt{e+fz^n}}$.

14. Given the latitude and longitude of a fixed star; Required the angle which the direction of the earth's way makes with a line drawn from the earth to the star, on a given day.

15. Find the time of the year at which a star whose right ascension and declination are known, rises heliacally, to an observer in a given latitude.

16. Shew that the force of the resistance on a sphere, moving in a fluid, with a given velocity, is to the force that would destroy the sphere's whole motion, in the time in which it would uniformly describe $\frac{8}{3}$ of its diameter, as the density of the fluid to that of the sphere.

17. Find the value of the disturbing force of S upon P, in the case of the three bodies (Sect. XI.); and deduce the mean quantity of that force, during one revolution of P round T; supposing that M = the force of S at the distance r.

18. If the density of a fluid be proportional to the compressing force, and its particles be attracted by a force varying inversely as the distance from the center; shew that, distances from the

center being taken in geometric progression, the corresponding densities will be in geometric progression.

19. Investigate Cotes's method of determining the length of an arc of the meridian, on the planisphere.

20. Shew that, in any position of the moon's nodes, the mean horary motion of the nodes, in one synodic revolution of the moon, is equal to half their horary motion when the moon is in syzygies.

21. Having given two distances of a comet, in its parabolic orbit, from the sun, and the angle included; deduce this proportion for determining the perihelion (which is here supposed to lie between those distances.)—The sum of the square roots of the distances is to the difference as the cotangent of the semi-sum of half the true anomalies to the tangent of the semi-difference of the same.

22. Apply Newton's general proposition, in the 8th Section, to the case of Cotes's three last spirals; point out the circumstances that determine the spiral; and deduce his construction of the elliptic spiral.

23. The major and minor axes of an ellipse are given; and a body begins to descend, from the extremity of the minor axis, towards the center of force in the focus, with the velocity in the curve at that point. Compare the time of descent to the focus with the time of revolving in the curve, from the same point to the nearest apside.

24. Suppose a sphere to move in a resisting medium; it is required to cut off a segment, by a plane perpendicular to the direction of its motion, so that the resistance on the remaining frustum may be three-fourths of that on the end of a cylinder circumscribing the sphere.

84 4 145ST2 53 005 BA 1 6562

Stanford University Libraries
3 6105 011 765 463

CECIL H. GREEN LIBRARY
STANFORD UNIVERSITY LIBRARIES
STANFORD, CALIFORNIA 94305-6004
(650) 723-1493
grncirc@sulmail.stanford.edu
All books are subject to recall.

DATE DUE

DEC 1 0 2003

OCT 29 2003 u